Generalized Ultrametric Semilattices of Linear Signals

Eleftherios Matsikoudis
Edward A. Lee

Electrical Engineering and Computer Sciences
University of California at Berkeley

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Eleftherios Matsikoudis   Edward A. Lee

Abstract

We consider certain spaces of linear signals equipped with a standard prefix relation and a suitably defined generalized distance function. We introduce a new class of abstract structures, which we call generalized ultrametric semilattices, and prove a representation theorem stating that every generalized ultrametric semilattice with a totally ordered distance set is isomorphic to a space of that kind. It follows that the formal definition of generalized ultrametric semilattices with totally ordered distance sets constitutes an axiomatization of the first-order theory of those spaces.

1 Introduction

A *signal* is a variation that conveys information (e.g., see [31], [18]). Mathematically, we represent such a variation as a partial function from some ordered set $\langle T, \leq_T \rangle$ to some non-empty set V (see Definition 2.1). The ordered set $\langle T, \leq_T \rangle$ need not be totally ordered. If it is, then we speak of a *linear* signal. The term "linear" here has nothing to do with algebra; it is used to indicate that the signal in question is defined over a set that is *linearly* ordered.

Signals are the bread and butter of electrical engineering. From power and control to electronics and telecommunications, they are ubiquitous in the mathematical modelling of systems. This is particularly true of linear signals: the totally ordered set $\langle T, \leq_T \rangle$ is used to model time, and a linear signal from $\langle T, \leq_T \rangle$ to V a variation in time. Historically, such signals have been of one of two types: *continuous-time* signals, where $\langle T, \leq_T \rangle$ is a continuum of values, typically the ordered set of all real numbers or that of all non-negative real numbers, and *discrete-time* signals, where $\langle T, \leq_T \rangle$ is a discrete set, typically the ordered set of all integers or that of all natural numbers. And in both cases, they have been total functions from $\langle T, \leq_T \rangle$ to V.

The need for signals that are partial functions from $\langle T, \leq_T \rangle$ to V emerges when considering the use of signals in the mathematical modelling of computation, an effort initiated in [17]. A notable example is that of a *discrete-event* signal, where $\langle T, \leq_T \rangle$ may be a continuum of values, but the domain of definition of the signal is only a discrete subset of T, and the intervals over which the signal is undefined are just as much a part of the conveyed information as the points at which the signal is defined. Such signals are essential to the mathematical modelling of timed computation (e.g., see [16]), and promise to provide the right interface between physical and computational processes in the emerging field of so-called "cyber-physical systems" (e.g., see [37], [8], [28]).

Once we allow for signals that are partial functions, we can define a very natural order relation on signals, namely the *prefix relation* on signals, which nicely organizes them into a semilattice (see Proposition 2.3). The prefix relation on signals yields a useful notion of approximation for signals, and grants access to the mathematical machinery of order theory.

There is, however, a different, yet equally important notion of approximation that stems from a natural, if abstract, notion of distance between any two signals, corresponding to the largest lower set of $\langle T, \leq_T \rangle$ over which the two signals agree; the larger the lower set, the closer the two signals. This abstract notion of distance gives rise to a *generalized distance function* on signals, which organizes them into a generalized ultrametric space (Proposition 2.5), and derives its importance from the fact that, in the mathematical modelling of timed computation, the causal functions are exactly the contracting functions, and the strictly causal functions are exactly the strictly contracting functions, with respect to that generalized distance function (see [27], [25]).

The prefix relation and the generalized distance function on signals are not independent of one another. Indeed, in [25], we proved that they satisfy two simple first-order properties of the following form:

1. $\forall \alpha_1 \ \forall \alpha_2 \ \forall \alpha_3 \ (\mathbf{d}(\alpha_1, \alpha_2) \leq \mathbf{d}(\alpha_1, \alpha_3) \rightarrow \alpha_1 \sqcap \alpha_3 \sqsubseteq \alpha_1 \sqcap \alpha_2)$;

2. $\forall \alpha_1 \ \forall \alpha_2 \ \forall \alpha_3 \ \mathbf{d}(\alpha_1 \sqcap \alpha_2, \alpha_1 \sqcap \alpha_3) \leq \mathbf{d}(\alpha_2, \alpha_3)$.

Here, $\forall$ is to be interpreted as universal quantification over signals from $\langle T, \leq_T \rangle$ to V, $\sqsubseteq$ as the prefix relation on those signals, $\sqcap$ as the associated meet operation, $\leq$ as inverse inclusion, and $\mathbf{d}$ as the generalized distance function on those signals (see Proposition 2.9).

The purpose of this work is twofold. First, we want to prove that, in fact, in the case of subsemilattices of linear signals, these two simple properties imply all formal properties of the relationship between the prefix relation and the generalized distance function. The restriction to subsemilattices is a natural one, but is also motivated by practical considerations (see [25, sec. 6]). For example, the set of all discrete-event signals from $\langle T, \leq_T \rangle$ to V forms a subsemilattice of the semilattice of all signals from $\langle T, \leq_T \rangle$ to V ordered by the prefix relation on those signals. And second, by proving that, we want to justify the introduction of a new class of abstract structures, which we call *generalized ultrametric semilattices*. These are, essentially, semilattices with a generalized ultrametric, or equivalently, generalized ultrametric spaces with a meet operation, and are formally defined as two sorted structures with a meet operation and a generalized ultrametric that satisfy the two properties in question.

Our interest in the relationship between the prefix relation and the generalized distance function on signals is motivated by the study of fixed-point semantics of timed systems. By and large, fixed-point semantics in computer science has been based on the fixed-point theory of order-preserving functions on ordered sets (see Tarski's fixed-point theorem and variants thereof), or that of contraction mappings on metric spaces (see Banach's contraction principle and variants thereof). In the case of timed systems, however, there are fixed-point problems involving strictly contracting functions on generalized ultrametric spaces of signals (see [30], [20]). And these problems are not amenable to standard classical methods, because there is no non-trivial order relation that will render every such function order-preserving (see [25, thm. A.2]), and no metric that will render every such function a contraction mapping (see [25, thm. A.4]). Until recently, the only tool available for dealing with such problems in a systematic way was a non-constructive fixed-point theorem of Priess-Crampe and Ribenboim for strictly contracting functions on spherically complete generalized ultrametric spaces (see [32, thm. 1]). But in [25], the relationship between the prefix relation and the generalized distance function on signals was used to prove a constructive fixed-point theorem for strictly contracting functions on generalized ultrametric spaces of signals, at the same time delivering an induction principle for proving properties of the constructed fixed-points (see also [27]). By proving that properties 1 and 2 above imply all formal properties of the relationship between the prefix relation and the generalized distance function on linear signals, not only do we enable formal reasoning about such signals at a higher level of abstraction, ignoring their low-level representation details, but we also allow for the fixed-point theory of [25] to be lifted into an abstract fixed-point theory (see [26]), readily applicable to other domains of interest (e.g., see Example 3.10, 3.11, and 3.12).

The main contributions of this paper are the following:

- we define generalized ultrametric semilattices (see Definition 3.7);

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- we introduce the notion of a *standard generalized ultrametric semilattice of signals*, which is the paradigmatic example of a generalized ultrametric semilattice (see page 11);

- we give examples of generalized ultrametric semilattices other than standard generalized ultrametric semilattices of signals (see Example 3.10, 3.11, and 3.12);

- we prove a representation theorem stating that every generalized ultrametric semilattice with a totally ordered distance set is isomorphic to a standard generalized ultrametric semilattice of linear signals (see Theorem 4.7);

- we use that representation theorem to prove that properties 1 and 2 above, along with the standard axioms for semilattices and generalized ultrametric spaces with totally ordered distance sets, constitute an axiomatization of the first-order theory of standard generalized ultrametric semilattices of linear signals (see Theorem 5.3);

- we show that this axiomatization does not generalize to standard generalized ultrametric semilattices of arbitrary signals; that is, properties 1 and 2 above, along with the standard axioms for semilattices and generalized ultrametric spaces with arbitrarily ordered distance sets, do not axiomatize the theory of standard generalized ultrametric semilattices of arbitrary signals (see Example 5.7).

This paper is an extended version of [24]. It provides the details missing from [24], includes examples of generalized ultrametric semilattices other than standard generalized ultrametric semilattices of signals (see Example 3.10, 3.11, and 3.12), shows that the axiomatization theorem does not generalize to standard generalized ultrametric semilattices of arbitrary signals (see Example 5.7), and addresses a potential criticism of the choice of structures taken into account in the representation and axiomatization theorems (see Proposition 4.8, Example 4.9, Theorem 5.5, and Corollary 5.6).

Our work should be contrasted with the various efforts aimed at the unification of the mathematical models based on ordered sets and metric spaces (see Section 6). To our knowledge, this is the first attempt at a systematic study of the relationship between order and distance in spaces that are naturally equipped with both.

The rest of this document is organized into six sections. In Section 2, we review the basics of signals, and examine the relationship between the prefix relation and the generalized distance function on signals. In Section 3, we define generalized ultrametric semilattices, study their basic model-theoretic properties, and give examples of generalized ultrametric semilattices, starting with the important class of standard generalized ultrametric semilattices of signals. In Section 4, we methodically prove our representation theorem for generalized ultrametric semilattices with totally ordered distance sets, and in Section 5, we obtain our sought axiomatization of the first-order theory of standard generalized ultrametric semilattices of linear signals. In Section 6, we discuss related work, and in Section 7, we conclude with a few comments on our results, and some directions for future work.

2 Signals[1]

Assume an ordered set[2] $\langle T, \leq_T \rangle$ and a non-empty set V.

Definition 2.1. A *signal* from $\langle T, \leq_T \rangle$ to V is a single-valued[3] subset of $T \times V$.

We write $\mathrm{S}[\langle T, \leq_T \rangle, V]$ for the set of all signals from $\langle T, \leq_T \rangle$ to V.

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[1] We refer to [25, sec. 2] for proofs for all propositions in this section.

[2] An *ordered set* is an ordered pair $\langle P, \leqslant \rangle$ such that P is a set, and $\leqslant$ is a reflexive, transitive, and antisymmetric binary relation on P.

[3] For every set A and B, and every $S \subseteq A \times B$, S is *single-valued* if and only if for any $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in S$, if $a_1 = a_2$, then $b_1 = b_2$.

Our concept of signal is based on [17], where members of T were referred to as *tags*, and members of V as *values*. But here, unlike in [17], we restrict signals to be single-valued.

Notice that the empty set is vacuously single-valued, and hence, by Definition 2.1, a signal from any ordered set to any non-empty set.

We call the empty set the *empty signal*.

We adopt common practice in modern set theory and identify a function with its graph. A signal from $\langle T, \leq_T \rangle$ to V is then a function with domain some subset of T, and range some subset of V, or in other words, a partial function from T to V.

Assume $s_1, s_2 \in S[\langle T, \leq_T \rangle, V]$ and $t \in T$.

We write $s_1(t) \simeq s_2(t)$ if and only if one of the following is true:

1. $t \notin \mathsf{dom}\, s_1$ and $t \notin \mathsf{dom}\, s_2$;
2. $t \in \mathsf{dom}\, s_1$, $t \in \mathsf{dom}\, s_2$, and $s_1(t) = s_2(t)$.

In other words, we use $\simeq$ to denote Kleene's equality among partially defined value expressions.

A special case of interest is when $\langle T, \leq_T \rangle$ is totally ordered.

We say that a signal from $\langle T, \leq_T \rangle$ to V is *linear* if and only if $\langle T, \leq_T \rangle$ is totally ordered.

It will at times be convenient to think of T as standing for some, possibly conceptual, notion of time, especially when $\langle T, \leq_T \rangle$ is totally ordered. The order relation $\leq_T$ will then play the role of a chronological precedence relation. But of course, this is but an interpretation, which is by no means the only possible one (e.g., see Example 3.12).

There is a natural order relation on signals, namely the prefix relation on signals.

We write $\sqsubseteq_{S[\langle T, \leq_T \rangle, V]}$ for a binary relation on $S[\langle T, \leq_T \rangle, V]$ such that for every $s_1, s_2 \in S[\langle T, \leq_T \rangle, V]$,

$$s_1 \sqsubseteq_{S[\langle T, \leq_T \rangle, V]} s_2 \iff \text{for every } t, t' \in T, \text{ if } t \in \mathsf{dom}\, s_1 \text{ and } t' \leq_T t, \text{ then } s_1(t') \simeq s_2(t').$$

Assume $s_1, s_2 \in S[\langle T, \leq_T \rangle, V]$.

We say that s_1 is a *prefix* of s_2 if and only if $s_1 \sqsubseteq_{S[\langle T, \leq_T \rangle, V]} s_2$.

Notice that for every $s \in S[\langle T, \leq_T \rangle, V]$, $\emptyset \sqsubseteq_{S[\langle T, \leq_T \rangle, V]} s$; that is, the empty signal is a prefix of every signal.

Proposition 2.2. $\langle S[\langle T, \leq_T \rangle, V], \sqsubseteq_{S[\langle T, \leq_T \rangle, V]} \rangle$ *is an ordered set.*

If $\langle T, \leq_T \rangle$ is totally ordered, then $\langle S[\langle T, \leq_T \rangle, V], \sqsubseteq_{S[\langle T, \leq_T \rangle, V]} \rangle$ has a tree-like structure. But in any case, every two signals in $S[\langle T, \leq_T \rangle, V]$ have a greatest common prefix in $S[\langle T, \leq_T \rangle, V]$.

Proposition 2.3. $\langle S[\langle T, \leq_T \rangle, V], \sqsubseteq_{S[\langle T, \leq_T \rangle, V]} \rangle$ *is a semilattice[4].*

In fact, $\langle S[\langle T, \leq_T \rangle, V], \sqsubseteq_{S[\langle T, \leq_T \rangle, V]} \rangle$ is a complete semilattice (see [25, prop. 2.11]), but we will not be interested in completeness properties here.

We write $\sqcap_{S[\langle T, \leq_T \rangle, V]}$ for a binary operation on $S[\langle T, \leq_T \rangle, V]$ such that for every $s_1, s_2 \in S[\langle T, \leq_T \rangle, V]$, $s_1 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_2$ is the greatest lower bound of s_1 and s_2 in $\langle S[\langle T, \leq_T \rangle, V], \sqsubseteq_{S[\langle T, \leq_T \rangle, V]} \rangle$.

The next proposition provides an alternative, and arguably, more intuitive definition of the prefix relation on signals.

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[4] An ordered set $\langle P, \leq \rangle$ is a *semilattice* (also called a *meet-semilattice* or a *lower semilattice*) if and only if for any $p_1, p_2 \in P$, there is a greatest lower bound (also called a *meet*) of p_1 and p_2 in $\langle P, \leq \rangle$.

Proposition 2.4. $s_1 \sqsubseteq s_2$ *if and only if there is* $L \in \mathscr{L} \langle T, \leq_T \rangle$ *such that* $s_1 = s_2 \restriction L$.[57]

There is also a natural, if abstract, notion of distance between any two signals, corresponding to the largest initial segment of the ordered set of tags, and over which the two signals agree; the larger the segment, the closer the two signals. Under certain conditions, this can be couched in the language of metric spaces (e.g., see [17], [16], [19]). All one needs is a map from such initial segments to non-negative real numbers. But this step of indirection excessively restricts the kind of ordered sets that one can use as tag sets (e.g., see [20]), and in fact, can be avoided as long as one is willing to think about the notion of distance in more abstract terms, and use the language of generalized ultrametric spaces[8] instead (see [33]).

We write $\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$ for a function from $\mathrm{S}[\langle T, \leq_T \rangle, V] \times \mathrm{S}[\langle T, \leq_T \rangle, V]$ to $\mathscr{L} \langle T, \leq_T \rangle$ such that for every $s_1, s_2 \in \mathrm{S}[\langle T, \leq_T \rangle, V]$,

$$\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_2) = \{t \mid t \in T, \text{ and for every } t' \leq_T t, \ s_1(t') \simeq s_2(t')\}.$$

Proposition 2.5. $\langle \mathrm{S}[\langle T, \leq_T \rangle, V], \mathscr{L} \langle T, \leq_T \rangle, \supseteq_{\mathscr{L} \langle T, \leq_T \rangle}, T, \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]} \rangle$ *is a generalized ultrametric space.*[10]

The following is immediate, and indeed, equivalent:

Proposition 2.6. *For every* $s_1, s_2, s_3 \in \mathrm{S}[\langle T, \leq_T \rangle, V]$, *the following are true:*

1. $\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_2) = T$ *if and only if* $s_1 = s_2$;
2. $\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_2) = \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_2, s_1)$;
3. $\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_2) \supseteq \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_3) \cap \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_3, s_2)$.

Actually, $\langle \mathrm{S}[\langle T, \leq_T \rangle, V], \mathscr{L} \langle T, \leq_T \rangle, \supseteq_{\mathscr{L} \langle T, \leq_T \rangle}, T, \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]} \rangle$ is a spherically complete generalized ultrametric space (see [21, lem. 2]), but again, we will not be concerned with completeness properties here.

If $\langle T, \leq_T \rangle$ is totally ordered, then $\langle \mathscr{L} \langle T, \leq_T \rangle, \supseteq_{\mathscr{L} \langle T, \leq_T \rangle} \rangle$ is also totally ordered. The following is an instance of the strict generalized ultrametric inequality[11], which is true in every generalized ultrametric space with a totally ordered distance set:

Proposition 2.7. *If* $\langle T, \leq_T \rangle$ *is totally ordered, then for every* $s_1, s_2, s_3 \in \mathrm{S}[\langle T, \leq_T \rangle, V]$ *and every* $L \in \mathscr{L} \langle T, \leq_T \rangle$, *if* $\mathrm{d}(s_1, s_2) \supset L$ *and* $\mathrm{d}(s_2, s_3) \supset L$, *then* $\mathrm{d}(s_1, s_3) \supset L$.

Note that the hypothesis of $\langle T, \leq_T \rangle$ being totally ordered in Proposition 2.7 cannot be discarded (see [25, exam. 2.8]).

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[5] For every ordered set $\langle P, \leqslant \rangle$, we write $\mathscr{L} \langle P, \leqslant \rangle$ for the set of all lower sets[6] of $\langle P, \leqslant \rangle$.

[6] For every ordered set $\langle P, \leqslant \rangle$, and every $L \subseteq P$, L is a *lower set* (also called a *down-set* or an *order ideal*) of $\langle P, \leqslant \rangle$ if and only if for any $p_1, p_2 \in P$, if $p_1 \leqslant p_2$ and $p_2 \in L$, then $p_1 \in L$.

[7] For every function f and every set A, we write $f \restriction A$ for the *restriction* of f to A, namely the function $\{\langle a, b \rangle \mid a \in A \text{ and } \langle a, b \rangle \in f\}$.

[8] A *generalized ultrametric space* is a quintuple $\langle A, P, \leqslant, 0, d \rangle$ such that A is a set, $\langle P, \leqslant, 0 \rangle$ is a pointed[9] ordered set, d is a function from $A \times A$ to P, and for any $a_1, a_2, a_3 \in A$ and every $p \in P$, the following are true:

1. $d(a_1, a_2) = 0$ if and only if $a_1 = a_2$;
2. $d(a_1, a_2) = d(a_2, a_1)$;
3. if $d(a_1, a_2) \leqslant p$ and $d(a_2, a_3) \leqslant p$, then $d(a_1, a_3) \leqslant p$.

We refer to clause 1 as the *identity of indiscernibles*, clause 2 as *symmetry*, and clause 3 as the *generalized ultrametric inequality*.

[9] An ordered set is *pointed* if and only if it has a least element. We write $\langle P, \leqslant, 0 \rangle$ for a pointed ordered set $\langle P, \leqslant \rangle$ with least element 0.

[10] For every set A, we write $\subseteq_A$ for the restriction of the inclusion class relation to A, namely the relation $\{\langle a_1, a_2 \rangle \mid a_1, a_2 \in A \text{ and } a_1 \subseteq a_2\}$, and $\supseteq_A$ for its inverse.

[11] For every generalized ultrametric space $\langle A, P, \leqslant, 0, d \rangle$, if $\langle P, \leqslant \rangle$ is totally ordered, then for any $a_1, a_2, a_3 \in A$, and every $p \in P$, if $d(a_1, a_2) < p$ and $d(a_2, a_3) < p$, then $d(a_1, a_3) < p$. We refer to this as the *strict generalized ultrametric inequality*.

Finally, we begin to probe the relationship between $\sqsubseteq_{S[\langle T, \leq_T\rangle, V]}$ and $d_{S[\langle T, \leq_T\rangle, V]}$.

Proposition 2.8. $s_1 \sqcap_{S[\langle T, \leq_T\rangle, V]} s_2 = s_1 \restriction d_{S[\langle T, \leq_T\rangle, V]}(s_1, s_2)$.

Proposition 2.8, which is a special case of a more general theorem (see [25, thm. 2.13]), portrays the relationship between $\sqsubseteq_{S[\langle T, \leq_T\rangle, V]}$ and $d_{S[\langle T, \leq_T\rangle, V]}$ in very concrete terms. Being expressed in the language of set theory, it is closely tied to the low-level representation of signals. In practice, one would rather work at a higher level of abstraction, and ignore the low-level representation details. The next proposition aims at distilling the essence of Proposition 2.8 into a couple of simple properties expressed in a language that only references $\sqsubseteq_{S[\langle T, \leq_T\rangle, V]}$ and $d_{S[\langle T, \leq_T\rangle, V]}$.[12]

Proposition 2.9. *For every $s_1, s_2, s_3 \in S[\langle T, \leq_T\rangle, V]$, the following are true:*

1. *if $d_{S[\langle T, \leq_T\rangle, V]}(s_1, s_2) \supseteq d_{S[\langle T, \leq_T\rangle, V]}(s_1, s_3)$, then $s_1 \sqcap_{S[\langle T, \leq_T\rangle, V]} s_3 \sqsubseteq_{S[\langle T, \leq_T\rangle, V]} s_1 \sqcap_{S[\langle T, \leq_T\rangle, V]} s_2$;*

2. *$d_{S[\langle T, \leq_T\rangle, V]}(s_1 \sqcap_{S[\langle T, \leq_T\rangle, V]} s_2, s_1 \sqcap_{S[\langle T, \leq_T\rangle, V]} s_3) \supseteq d_{S[\langle T, \leq_T\rangle, V]}(s_2, s_3)$.*

Figure 1 is an attempt at a visualization of Proposition 2.9 in the case where $\langle T, \leq_T\rangle$ is totally ordered. Signals are pictured as arrows, all emanating from the same, leftmost point of each diagram. A signal is a prefix of another signal just as long as the arrow corresponding to the former is an initial segment of the arrow corresponding to the latter. The distance between two signals is represented by the point where the corresponding arrows diverge from one another; the more to the right the point of divergence, the smaller the distance between the two signals. An exception would be the case of a distance between a signal and a prefix of that signal, which would be represented by a point that lied at least as much to the right as the right end of the arrow corresponding to the prefix signal, depending on the particular choice of signals. Instances of that case have been omitted from Figure 1. For the sake of clarity, the subscripts of $\sqcap_{S[\langle T, \leq_T\rangle, V]}$ and $d_{S[\langle T, \leq_T\rangle, V]}$ have been dropped.

The four diagrams of Figure 1 illustrate the four non-trivial cases where all three signals are different from one another. For example, in the diagram of Figure 1(a), where the distance between s_1 and s_2 is strictly smaller than the distance between s_1 and s_3, the meet of s_1 and s_3 is indeed a prefix of the meet of s_1 and s_2. And in that same diagram, the distance between the meet of s_1 and s_2 and the meet of s_1 and s_3 is at least as small as the distance between s_2 and s_3.

Parenthetically, we note that "arrow-divergence" diagrams of this kind, while useful aids to intuition in the case where $\langle T, \leq_T\rangle$ is totally ordered, can be quite misleading in the case where $\langle T, \leq_T\rangle$ is partially ordered.

We first came across the two properties of Proposition 2.9 while first proving the main fixed-point theorem of [25]; they emerged as a minimal set of postulates sufficient to eliminate any reference to individual tags and values. Determining the extent to which they succeed in capturing the relationship between $\sqsubseteq_{S[\langle T, \leq_T\rangle, V]}$ and $d_{S[\langle T, \leq_T\rangle, V]}$ is the subject of this work.

Proposition 2.9.1 is actually true in every semilattice of signals (see [25, proof of prop. 2.15.1]). This is not the case for Proposition 2.9.2.

Example 2.10. Let $T = \{0, 1, 2\}$, and $\leq_T$ be the standard order on $\{0, 1, 2\}$.

Let V be a non-empty set, and v a member of V.

Let $s_1 = \{\langle 0, v\rangle, \langle 1, v\rangle\}$.

Let $s_2 = \{\langle 0, v\rangle, \langle 2, v\rangle\}$.

Let $s_3 = \emptyset$.

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[12] Notice that $\sqcap_{S[\langle T, \leq_T\rangle, V]}$ is definable in $\langle S[\langle T, \leq_T\rangle, V], \sqsubseteq_{S[\langle T, \leq_T\rangle, V]}\rangle$, and conversely, $\sqsubseteq_{S[\langle T, \leq_T\rangle, V]}$ is definable in $\langle S[\langle T, \leq_T\rangle, V], \sqcap_{S[\langle T, \leq_T\rangle, V]}\rangle$ (see also footnote 16 and comment following Definition 3.7).
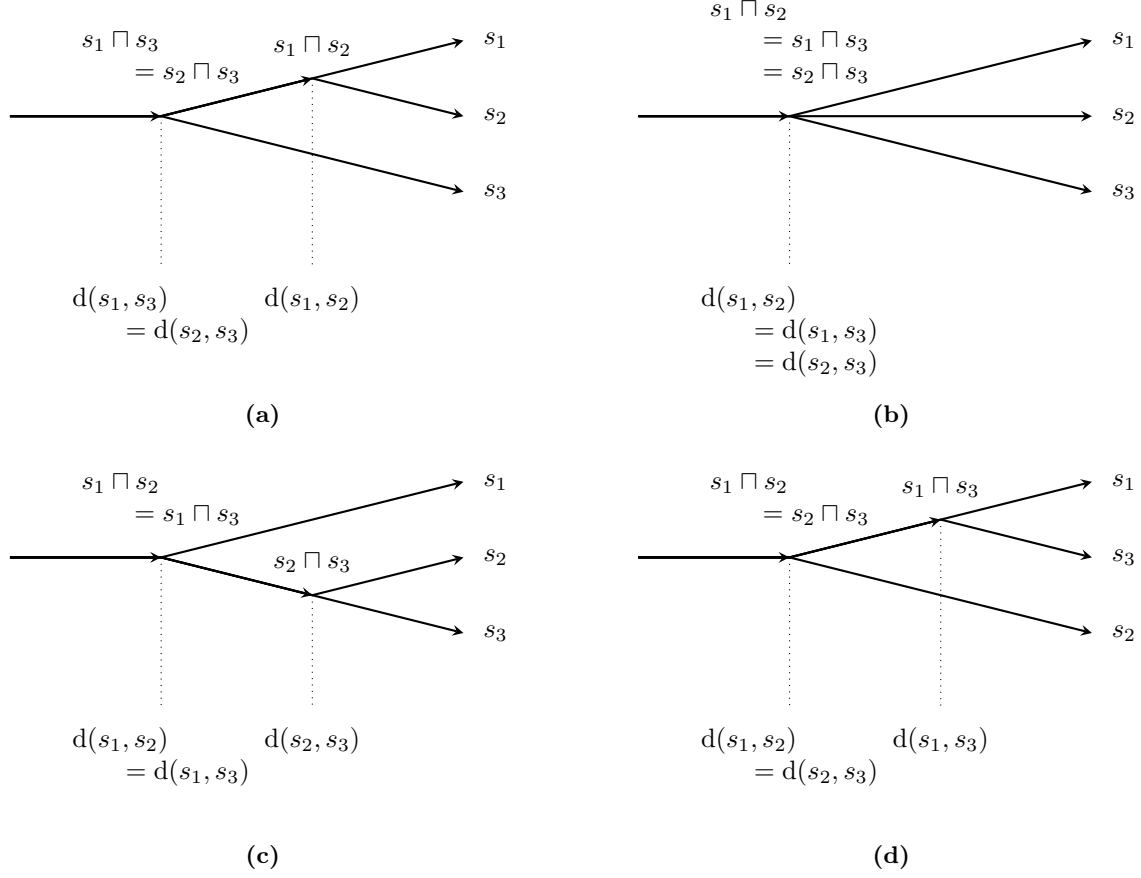
(a)

$s_1 \sqcap s_3$
$= s_2 \sqcap s_3$ $s_1 \sqcap s_2$ s_1

s_2

s_3

$\mathrm{d}(s_1, s_3)$ $\mathrm{d}(s_1, s_2)$
$= \mathrm{d}(s_2, s_3)$

(b)

$s_1 \sqcap s_2$
$= s_1 \sqcap s_3$
$= s_2 \sqcap s_3$ s_1

s_2

s_3

$\mathrm{d}(s_1, s_2)$
$= \mathrm{d}(s_1, s_3)$
$= \mathrm{d}(s_2, s_3)$

(c)

$s_1 \sqcap s_2$
$= s_1 \sqcap s_3$ s_1

$s_2 \sqcap s_3$ s_2

s_3

$\mathrm{d}(s_1, s_2)$ $\mathrm{d}(s_2, s_3)$
$= \mathrm{d}(s_1, s_3)$

(d)

$s_1 \sqcap s_2$ $s_1 \sqcap s_3$ s_1
$= s_2 \sqcap s_3$

s_3

s_2

$\mathrm{d}(s_1, s_2)$ $\mathrm{d}(s_1, s_3)$
$= \mathrm{d}(s_2, s_3)$

Figure 1. An attempt at a visualization of Proposition 2.9 in the case where $\langle T, \leq_T \rangle$ is totally ordered. Signals are pictured as arrows, all emanating from the same, leftmost point of each diagram. A signal is a prefix of another signal just as long as the arrow corresponding to the former is an initial segment of the arrow corresponding to the latter. The distance between two signals is represented by the point where the corresponding arrows diverge from one another; the more to the right the point of divergence, the smaller the distance between the two signals. For the sake of clarity, the subscripts of $\sqcap_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$ and $\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$ have been dropped.

Let $\sqsubseteq_{\{s_1, s_2, s_3\}}$ be the restriction[13] of $\sqsubseteq_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$ to $\{s_1, s_2, s_3\}$.

Clearly, $\langle \{s_1, s_2, s_3\}, \sqsubseteq_{\{s_1, s_2, s_3\}} \rangle$ is a semilattice.

Let $\sqcap_{\{s_1, s_2, s_3\}}$ be a binary operation on $\{s_1, s_2, s_3\}$ such that for every $s_1', s_2' \in \{s_1, s_2, s_3\}$, $s_1' \sqcap_{\{s_1, s_2, s_3\}} s_2'$ is the greatest lower bound of s_1' and s_2' in $\langle \{s_1, s_2, s_3\}, \sqsubseteq_{\{s_1, s_2, s_3\}} \rangle$.

Then

$$
\begin{aligned}
\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1 \sqcap_{\{s_1, s_2, s_3\}} s_1, s_1 \sqcap_{\{s_1, s_2, s_3\}} s_2) &= \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_3) \\
&= \emptyset \\
&\not\supseteq \{0\} \\
&= \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_2).
\end{aligned}
$$

---

[13] For every binary relation R and every set A, the *restriction* of R to A is the relation $\{\langle a_1, a_2 \rangle \mid a_1, a_2 \in A \text{ and } \langle a_1, a_2 \rangle \in R\}$.

However, for every semilattice of signals from $\langle T, \leq_T \rangle$ to V, if that semilattice is a subsemilattice[14] of $\langle \mathrm{S}[\langle T, \leq_T \rangle, V], \sqsubseteq_{\mathrm{S}[\langle T, \leq_T \rangle, V]} \rangle$, then both clauses of Proposition 2.9 are true in it. Rather pleasingly, the converse is also true.

Assume $X \subseteq \mathrm{S}[\langle T, \leq_T \rangle, V]$.

We write $\sqsubseteq_X$ for the restriction of $\sqsubseteq_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$ to X.

If $\langle X, \sqsubseteq_X \rangle$ is a semilattice, then we write $\sqcap_X$ for a binary operation on X such that for every $s_1, s_2 \in X$, $s_1 \sqcap_X s_2$ is the greatest lower bound of s_1 and s_2 in $\langle X, \sqsubseteq_X \rangle$.

Proposition 2.11. *If $\langle X, \sqsubseteq_X \rangle$ is a semilattice, then the following are equivalent:*

1. *for every $s_1, s_2, s_3 \in X$, the following are true:*

    (a) *if $\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_2) \supseteq \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_3)$, then $s_1 \sqcap_X s_3 \sqsubseteq_{\mathrm{S}[\langle T, \leq_T \rangle, V]} s_1 \sqcap_X s_2$;*

    (b) *$\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1 \sqcap_X s_2, s_1 \sqcap_X s_3) \supseteq \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_2, s_3)$;*

2. *$\langle X, \sqsubseteq_X \rangle$ is a subsemilattice of $\langle \mathrm{S}[\langle T, \leq_T \rangle, V], \sqsubseteq_{\mathrm{S}[\langle T, \leq_T \rangle, V]} \rangle$.*

3 Generalized ultrametric semilattices

In order to appreciate the significance of Proposition 2.9, we introduce a new kind of abstract structure with a meet operation and a generalized distance function that satisfy the two clauses of Proposition 2.9. And although it is natural to think of structures of this kind as ordered generalized ultrametric spaces, we find it more convenient to strip generalized distances of their distinguished status, and treat such spaces as two-sorted structures. We assume that the reader is familiar with the concept of many-sorted signature, which is, of course, a straightforward generalization of that in the one-sorted case (e.g., see [12, chap. 1.1]).

We write Σ for a two-sorted signature consisting of two sorts A and D, and the following symbols:

1. an infix function symbol $\sqcap$ of type $\mathrm{A} \times \mathrm{A} \to \mathrm{A}$;

2. an infix relation symbol $\leq$ of type $\mathrm{D} \times \mathrm{D}$;

3. a constant symbol $\mathbf{0}$ of type $\mathbf{1} \to \mathrm{D}$;

4. a function symbol $\mathbf{d}$ of type $\mathrm{A} \times \mathrm{A} \to \mathrm{D}$.

Definition 3.1. A Σ-*structure* is a function $\mathfrak{A}$ from the set of sorts and symbols of Σ such that $\mathfrak{A}(\mathrm{A})$ and $\mathfrak{A}(\mathrm{D})$ are non-empty sets, and the following are true:

1. $\mathfrak{A}(\sqcap)$ is a function from $\mathfrak{A}(\mathrm{A}) \times \mathfrak{A}(\mathrm{A})$ to $\mathfrak{A}(\mathrm{A})$;

2. $\mathfrak{A}(\leq)$ is a subset of $\mathfrak{A}(\mathrm{D}) \times \mathfrak{A}(\mathrm{D})$;

3. $\mathfrak{A}(\mathbf{0})$ is a member of $\mathfrak{A}(\mathrm{D})$;

4. $\mathfrak{A}(\mathbf{d})$ is a function from $\mathfrak{A}(\mathrm{A}) \times \mathfrak{A}(\mathrm{A})$ to $\mathfrak{A}(\mathrm{D})$.

Assume a Σ-structure $\mathfrak{A}$.

We write $|\mathfrak{A}|_{\mathrm{A}}$ for $\mathfrak{A}(\mathrm{A})$, $|\mathfrak{A}|_{\mathrm{D}}$ for $\mathfrak{A}(\mathrm{D})$, $\sqcap^{\mathfrak{A}}$ for $\mathfrak{A}(\sqcap)$, $\leq^{\mathfrak{A}}$ for $\mathfrak{A}(\leq)$, $\mathbf{0}^{\mathfrak{A}}$ for $\mathfrak{A}(\mathbf{0})$, and $\mathbf{d}^{\mathfrak{A}}$ for $\mathfrak{A}(\mathbf{d})$.

We call $|\mathfrak{A}|_{\mathrm{A}}$ the *carrier* of $\mathfrak{A}$ of sort A, or the *abstract set* of $\mathfrak{A}$, and $|\mathfrak{A}|_{\mathrm{D}}$ the *carrier* of $\mathfrak{A}$ of sort D, or the *distance set* of $\mathfrak{A}$.

---

[14] For every semilattice $\langle P, \leqslant \rangle$, and every $S \subseteq P$, $\langle S, \leqslant \rangle$ is a *subsemilattice* of $\langle P, \leqslant \rangle$ if and only if for every $p_1, p_2 \in S$, there is a greatest lower bound of p_1 and p_2 in $\langle S, \leqslant \rangle$, and that greatest lower bound is the greatest lower bound of p_1 and p_2 in $\langle P, \leqslant \rangle$.

Assume Σ-structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Definition 3.2. A *homomorphism* from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ is an $\{A, D\}$-indexed family $\{h_A, h_D\}$ of a function h_A from $|\mathfrak{A}_1|_A$ to $|\mathfrak{A}_2|_A$ and a function h_D from $|\mathfrak{A}_1|_D$ to $|\mathfrak{A}_2|_D$ such that the following are true:

1. for every $a_1, a_2 \in |\mathfrak{A}_1|_A$, $h_A(a_1 \sqcap^{\mathfrak{A}_1} a_2) = h_A(a_1) \sqcap^{\mathfrak{A}_2} h_A(a_2)$;

2. for every $d_1, d_2 \in |\mathfrak{A}_1|_D$, if $d_1 \leq^{\mathfrak{A}_1} d_2$, then $h_D(d_1) \leq^{\mathfrak{A}_2} h_D(d_2)$;

3. $h_D(\mathbf{0}^{\mathfrak{A}_1}) = \mathbf{0}^{\mathfrak{A}_2}$;

4. for every $a_1, a_2 \in |\mathfrak{A}_1|_A$, $h_D(\mathbf{d}^{\mathfrak{A}_1}(a_1, a_2)) = \mathbf{d}^{\mathfrak{A}_2}(h_A(a_1), h_A(a_2))$.

We say that $\{h_A, h_D\}$ is an *embedding* of $\mathfrak{A}_1$ into $\mathfrak{A}_2$ if and only if $\{h_A, h_D\}$ is a homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$, h_A is one-to-one, h_D is one-to-one, and for every $d_1, d_2 \in |\mathfrak{A}_1|_D$, $d_1 \leq^{\mathfrak{A}_1} d_2$ if and only if $h_D(d_1) \leq^{\mathfrak{A}_2} h_D(d_2)$.

We say that $\{h_A, h_D\}$ is an *isomorphism* between $\mathfrak{A}_1$ and $\mathfrak{A}_2$ if and only if $\{h_A, h_D\}$ is a homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$, h_A is a one-to-one correspondence between $|\mathfrak{A}_1|_A$ and $|\mathfrak{A}_2|_A$, h_D is a one-to-one correspondence between $|\mathfrak{A}_1|_D$ and $|\mathfrak{A}_2|_D$, and for every $d_1, d_2 \in |\mathfrak{A}_1|_D$, $d_1 \leq^{\mathfrak{A}_1} d_2$ if and only if $h_D(d_1) \leq^{\mathfrak{A}_2} h_D(d_2)$.

Proposition 3.3. *If $\{h_A, h_D\}$ is an* isomorphism *between $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then $\{h_A^{-1}, h_D^{-1}\}$ is an* isomorphism *between $\mathfrak{A}_2$ and $\mathfrak{A}_1$.*

We say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *isomorphic* if and only if there is an isomorphism between $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Definition 3.4. A *substructure* of $\mathfrak{A}$ is a Σ-structure $\mathfrak{A}'$ such that $|\mathfrak{A}'|_A \subseteq |\mathfrak{A}|_A$, $|\mathfrak{A}'|_D \subseteq |\mathfrak{A}|_D$, and $\{|\mathfrak{A}'|_A \hookrightarrow |\mathfrak{A}|_A, |\mathfrak{A}'|_D \hookrightarrow |\mathfrak{A}|_D\}$ is an embedding of $\mathfrak{A}'$ into $\mathfrak{A}$.[15]

We write $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ if and only if $\mathfrak{A}_1$ is a substructure of $\mathfrak{A}_2$.

For every class $\mathcal{C}$ of Σ-structures, we write $\mathsf{S}(\mathcal{C})$ for the class $\{\mathfrak{A} \mid$ there is $\mathfrak{A}'$ in $\mathcal{C}$ such that $\mathfrak{A} \subseteq \mathfrak{A}'\}$.

Proposition 3.5. *If $\{h_A, h_D\}$ is an embedding of $\mathfrak{A}_1$ into $\mathfrak{A}_2$, then there is a substructure $\mathfrak{A}_2'$ of $\mathfrak{A}_2$ such that $\{h_A, h_D\}$ is an isomorphism between $\mathfrak{A}_1$ and $\mathfrak{A}_2'$.*

The concepts of homomorphism, embedding, isomorphism, and substructure for Σ-structures are instances of the standard concepts of homomorphism, embedding, isomorphism, and substructure for many-sorted structures, which are, of course, straightforward generalizations of those for one-sorted structures (e.g., see [12, chap. 1.2]).

For reasons that will become clear, we shall also be interested in substructures whose distance sets are the same as those of the original structures.

We say that $\mathfrak{A}'$ is an A-substructure of $\mathfrak{A}$ if and only if $\mathfrak{A}'$ is a substructure of $\mathfrak{A}$, and $|\mathfrak{A}'|_D = |\mathfrak{A}|_D$.

We write $\mathfrak{A}_1 \subseteq_A \mathfrak{A}_2$ if and only if $\mathfrak{A}_1$ is an A-substructure of $\mathfrak{A}_2$.

For every class $\mathcal{C}$ of Σ-structures we write $\mathsf{S}_A(\mathcal{C})$ for the class $\{\mathfrak{A} \mid$ there is $\mathfrak{A}'$ in $\mathcal{C}$ such that $\mathfrak{A} \subseteq_A \mathfrak{A}'\}$.

The following is immediate from Proposition 3.5:

Proposition 3.6. *If $\{h_A, h_D\}$ is an embedding of $\mathfrak{A}_1$ into $\mathfrak{A}_2$, then there is an A-substructure $\mathfrak{A}_2'$ of $\mathfrak{A}_2$ such that $\{h_A, h_D\}$ is an embedding of $\mathfrak{A}_1$ into $\mathfrak{A}_2'$, and h_A is a one-to-one correspondence between $|\mathfrak{A}_1|_A$ and $|\mathfrak{A}_2'|_A$.*

---

[15] For every set S_1 and S_2 such that $S_1 \subseteq S_2$, we write $S_1 \hookrightarrow S_2$ for a function from S_1 to S_2 such that for any $s_1 \in S_1$, $(S_1 \hookrightarrow S_2)(s_1) = s_1$. We call $S_1 \hookrightarrow S_2$ the *inclusion map* from S_1 to S_2.

Now, the Σ-structures that we are interested in are those in which the function assigned to $\sqcap$ behaves as the meet operation of a semilattice, the function assigned to $\mathbf{d}$ as the generalized distance function of a generalized ultrametric space, and the two satisfy the two clauses of Proposition 2.9.

Definition 3.7. A *generalized ultrametric semilattice* is a Σ-structure $\mathfrak{A}$ such that the following are true:

1. $\langle |\mathfrak{A}|_{\mathrm{A}}, \sqcap^{\mathfrak{A}} \rangle$ is a semilattice[16];

2. $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}} \rangle$ is a pointed ordered set;

3. $\langle |\mathfrak{A}|_{\mathrm{A}}, |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}}, \mathbf{d}^{\mathfrak{A}} \rangle$ is a generalized ultrametric space;

4. for every $a_1, a_2, a_3 \in |\mathfrak{A}|_{\mathrm{A}}$, the following are true:

    (a) if $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_3)$, then $(a_1 \sqcap^{\mathfrak{A}} a_3) \sqcap^{\mathfrak{A}} (a_1 \sqcap^{\mathfrak{A}} a_2) = a_1 \sqcap^{\mathfrak{A}} a_3$;

    (b) $\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1 \sqcap^{\mathfrak{A}} a_3) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_2, a_3)$.

An interesting thing to notice is that, in Section 2, a semilattice was viewed as an ordered set, whereas here, it is viewed as an algebraic structure. The two views are closely connected, and one may seamlessly switch between them (e.g., see [4, lem. 2.8]). But formally, it will be simpler to work with a meet operation than with an order relation. And informally, we will recover the order relation from the meet operation, and for every $a_1, a_2 \in |\mathfrak{A}|_{\mathrm{A}}$, write $a_1 \sqsubseteq^{\mathfrak{A}} a_2$ if and only if $a_1 \sqcap^{\mathfrak{A}} a_2 = a_1$. In particular, we may rewrite Definition 3.7.4 in the form of Proposition 2.9:

4. for every $a_1, a_2, a_3 \in |\mathfrak{A}|_{\mathrm{A}}$, the following are true:

    (a) if $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_3)$, then $a_1 \sqcap^{\mathfrak{A}} a_3 \sqsubseteq^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_2$;

    (b) $\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1 \sqcap^{\mathfrak{A}} a_3) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_2, a_3)$.

Of course, all this can be done formally, but we shall not worry ourselves over the details.

The following is straightforward:

Proposition 3.8. *If $\mathfrak{A}'$ is a substructure of $\mathfrak{A}$, and $\mathfrak{A}$ is a generalized ultrametric semilattice, then $\mathfrak{A}'$ is a generalized ultrametric semilattice.*

Clearly, $\sqcap_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$ and $\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$ structure $\mathrm{S}[\langle T, \leq_T \rangle, V]$ into a generalized ultrametric semilattice.

We write $\mathfrak{S}[\langle T, \leq_T \rangle, V]$ for a Σ-structure such that $|\mathfrak{S}[\langle T, \leq_T \rangle, V]|_{\mathrm{A}} = \mathrm{S}[\langle T, \leq_T \rangle, V]$, $|\mathfrak{S}[\langle T, \leq_T \rangle, V]|_{\mathrm{D}} = \mathscr{L}\langle T, \leq_T \rangle$, and the following are true:

1. $\sqcap^{\mathfrak{S}[\langle T, \leq_T \rangle, V]} = \sqcap_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$;

2. $\leq^{\mathfrak{S}[\langle T, \leq_T \rangle, V]} = \supseteq_{\mathscr{L}\langle T, \leq_T \rangle}$;

3. $\mathbf{0}^{\mathfrak{S}[\langle T, \leq_T \rangle, V]} = T$;

4. $\mathbf{d}^{\mathfrak{S}[\langle T, \leq_T \rangle, V]} = \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}$.

The following is immediate from Proposition 2.3, 2.5, and 2.9:

Proposition 3.9. $\mathfrak{S}[\langle T, \leq_T \rangle, V]$ *is a generalized ultrametric semilattice.*

---

[16] For every set S, and every binary operation $\sqcap$ on S, $\langle S, \sqcap \rangle$ is a *semilattice* if and only if for any $s_1, s_2, s_3 \in S$, the following are true:

1. $(s_1 \sqcap s_2) \sqcap s_3 = s_1 \sqcap (s_2 \sqcap s_3)$;

2. $s_1 \sqcap s_2 = s_2 \sqcap s_1$;

3. $s_1 \sqcap s_1 = s_1$.

We refer to $\mathfrak{S}[\langle T, \leq_T \rangle, V]$ as the *standard generalized ultrametric semilattice of all signals from* $\langle T, \leq_T \rangle$ *to* V.

We write $\mathcal{S}$ for the class $\{\mathfrak{S}[\langle T, \leq_T \rangle, V] \mid \langle T, \leq_T \rangle$ is an ordered set and V is a non-empty set$\}$.

We refer to every structure in $\mathsf{S}(\mathcal{S})$ as a *standard generalized ultrametric semilattice of signals*.

In this work, our main interest is in structures of linear signals.

We write $\mathcal{S}_{\mathrm{lin}}$ for the class $\{\mathfrak{S}[\langle T, \leq_T \rangle, V] \mid \langle T, \leq_T \rangle$ is a totally ordered set and V is a non-empty set$\}$.

We refer to every structure in $\mathsf{S}(\mathcal{S}_{\mathrm{lin}})$ as a *standard generalized ultrametric semilattice of linear signals*.

Standard generalized ultrametric semilattices of signals, and in particular, of linear signals, have direct application to the study of timed computation (see [25]).

An example of a non-standard generalized ultrametric semilattice of linear signals is the set of all finite and infinite sequences over some non-empty set of values, equipped with the standard prefix relation and the so-called "Baire-distance function" (e.g., see [5]).

Example 3.10. Let V be a non-empty set.

Let $\mathfrak{A}$ be a Σ-structure such that $|\mathfrak{A}|_{\mathrm{A}}$ is the set of all finite and infinite sequences over V, $|\mathfrak{A}|_{\mathrm{D}} = \mathbb{R}_{\geq 0}$,[17] and the following are true:

1. $\sqcap^{\mathfrak{A}}$ is a binary operation on $|\mathfrak{A}|_{\mathrm{A}}$ such that for every $s_1, s_2 \in |\mathfrak{A}|_{\mathrm{A}}$, $s_1 \sqcap^{\mathfrak{A}} s_2$ is the greatest common prefix of s_1 and s_2;

2. $\leq^{\mathfrak{A}}$ is the standard order on $\mathbb{R}_{\geq 0}$;

3. $\mathbf{0}^{\mathfrak{A}} = 0$;

4. $\mathbf{d}^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|_{\mathrm{A}} \times |\mathfrak{A}|_{\mathrm{A}}$ to $|\mathfrak{A}|_{\mathrm{D}}$ such that for every $s_1, s_2 \in |\mathfrak{A}|_{\mathrm{A}}$,

$$\mathbf{d}^{\mathfrak{A}}(s_1, s_2) = \begin{cases} 0 & \text{if } s_1 = s_2; \\ 2^{-\min\{n \mid n \in \mathbb{N} \text{ and } s_1(n) \not\simeq s_2(n)\}} & \text{otherwise.}^{18} \end{cases}$$

Clearly, for every $s_1, s_2, s_3, s_4 \in |\mathfrak{A}|_{\mathrm{A}}$,

$$\mathbf{d}^{\mathfrak{A}}(s_1, s_2) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(s_3, s_4)$$

if and only if

$$\mathrm{d}_{\mathsf{S}[\langle \mathbb{N}, \leq_{\mathbb{N}} \rangle, V]}(s_1, s_2) \supseteq \mathrm{d}_{\mathsf{S}[\langle \mathbb{N}, \leq_{\mathbb{N}} \rangle, V]}(s_3, s_4),$$

and thus, $\mathfrak{A}$ is a generalized ultrametric semilattice.

Notice that the generalized ultrametric space associated with the generalized ultrametric semilattice $\mathfrak{A}$ of Example 3.10 is a standard ultrametric space. In such a case, we may omit the term "generalized", and speak simply of an *ultrametric semilattice*.

Another example of a non-standard ultrametric semilattice of linear signals, one that is of particular interest to the study of timed computation, is the set of all discrete-event[19] real-time signals over some non-empty set of values, equipped with the standard prefix relation and the so-called "Cantor metric" (e.g., see [17], [16]).

---

[17] We write $\mathbb{R}_{\geq 0}$ for the set of all non-negative real numbers.

[18] We write $\mathbb{N}$ for the set of all natural numbers, and $\leq_{\mathbb{N}}$ for the standard order on $\mathbb{N}$.

[19] A signal s from $\langle T, \leq_T \rangle$ to V is *discrete-event* if and only if there is an order-embedding of $\langle \mathsf{dom}\, s, \leq_{\mathsf{dom}\, s} \rangle$ into $\langle \mathbb{N}, \leq_{\mathbb{N}} \rangle$, where $\leq_{\mathsf{dom}\, s}$ is the restriction of $\leq_T$ to $\mathsf{dom}\, s$.

Example 3.11. Let V be a non-empty set.

Let $\mathfrak{A}$ be a Σ-structure such that $|\mathfrak{A}|_A$ is the set of all discrete-event signals from $\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle$ to V,[20] $|\mathfrak{A}|_D = \mathbb{R}_{\geq 0}$, and the following are true:

1. $\sqcap^{\mathfrak{A}}$ is a binary operation on $|\mathfrak{A}|_A$ such that for every $s_1, s_2 \in |\mathfrak{A}|_A$, $s_1 \sqcap^{\mathfrak{A}} s_2$ is the greatest common prefix of s_1 and s_2;

2. $\leq^{\mathfrak{A}}$ is the standard order on $\mathbb{R}_{\geq 0}$;

3. $\mathbf{0}^{\mathfrak{A}} = 0$;

4. $\mathbf{d}^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|_A \times |\mathfrak{A}|_A$ to $|\mathfrak{A}|_D$ such that for every $s_1, s_2 \in |\mathfrak{A}|_A$,

$$\mathbf{d}^{\mathfrak{A}}(s_1, s_2) = \begin{cases} 0 & \text{if } s_1 = s_2; \\ 2^{-\min\{r | r \in \mathbb{R} \text{ and } s_1(r) \not\simeq s_2(r)\}} & \text{otherwise.} \end{cases}$$

Notice that since the domain of every signal in $|\mathfrak{A}|_A$ is well ordered by $\leq_{\mathbb{R}}$, for every $s_1, s_2 \in |\mathfrak{A}|_A$, $\{r \mid r \in \mathbb{R} \text{ and } s_1(r) \not\simeq s_2(r)\}$ is also well ordered by $\leq_{\mathbb{R}}$, and thus, $\min\{r \mid r \in \mathbb{R} \text{ and } s_1(r) \not\simeq s_2(r)\}$ is well defined.

It is easy to verify that, since the domain of every signal in $|\mathfrak{A}|_A$ is well ordered by $\leq_{\mathbb{R}}$, for every $s_1, s_2, s_3, s_4 \in |\mathfrak{A}|_A$,

$$\mathbf{d}^{\mathfrak{A}}(s_1, s_2) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(s_3, s_4)$$

if and only if

$$\mathrm{d}_{\mathrm{S}[\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle, V]}(s_1, s_2) \supseteq \mathrm{d}_{\mathrm{S}[\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle, V]}(s_3, s_4),$$

and thus, $\mathfrak{A}$ is an ultrametric semilattice. In fact, there is an embedding of $\mathfrak{A}$ into $\mathfrak{S}[\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle, V]$, as the reader may wish to verify.

It is natural to expect that, despite the extra step of indirection involved in its definition, it being a standard ultrametric, the Cantor metric on discrete-event real-time signals, namely $\mathbf{d}^{\mathfrak{A}}$ of Example 3.11, will be more popular than the more exotic $\mathrm{d}_{\mathrm{S}[\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle, V]}$ among those studying timed computation. It is, therefore, convenient that the ultrametric spaces already used for modelling timed systems are, in fact, ultrametric semilattices. As a consequence, the fixed-point theory of [26] can be directly applied to study the behaviour of strictly causal discrete-event components, modelled as strictly contracting functions on discrete-event signals, in feedback.

Finally, we include an example from the field of logic programming. We assume familiarity with the basic concepts of logic programming (e.g., see [22]). Our notation is based on [11].

Example 3.12. Let P be a normal logic program.

Let α be a non-empty countable ordinal, and l a function from H_P, the Herbrand base of P, to α.

Let $\mathfrak{A}$ be a Σ-structure such that $|\mathfrak{A}|_A$ is the set of all subsets of H_P, $|\mathfrak{A}|_D = \alpha \cup \{\alpha\}$, and the following are true:

1. $\sqcap^{\mathfrak{A}}$ is a binary operation on $|\mathfrak{A}|_A$ such that for every $I_1, I_2 \in |\mathfrak{A}|_A$,

$$I_1 \sqcap^{\mathfrak{A}} I_2 = \{A \mid A \in I_1, A \in I_2, \text{ and for every } A' \text{ such that } l(A') \in l(A) \text{ or } l(A') = l(A), A' \in I_1 \\ \text{if and only if } A' \in I_2\};$$

---

[20] We write $\mathbb{R}$ for the set of all real numbers, and $\leq_{\mathbb{R}}$ for the standard order on $\mathbb{R}$.

2. $\leq^{\mathfrak{A}}$ is a binary relation on $|\mathfrak{A}|_{\mathrm{D}}$ such that for every $\beta, \gamma \in |\mathfrak{A}|_{\mathrm{D}}$,

$$\beta \leq^{\mathfrak{A}} \gamma \iff \gamma \in \beta \text{ or } \beta = \gamma.$$

3. $\mathbf{0}^{\mathfrak{A}} = \alpha$;

4. $\mathbf{d}^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|_{\mathrm{A}} \times |\mathfrak{A}|_{\mathrm{A}}$ to $|\mathfrak{A}|_{\mathrm{D}}$ such that for every $I_1, I_2 \in |\mathfrak{A}|_{\mathrm{A}}$,

$$\mathbf{d}^{\mathfrak{A}}(I_1, I_2) = \{\beta \mid \beta \in \alpha, \text{ and for every } A \text{ such that } l(A') \in \beta \text{ or } l(A') = \beta, A' \in I_1 \text{ if and only if } A' \in I_2\}.$$

Let $\leq_a$ be a binary relation on α such that for every $\beta, \gamma \in \alpha$,

$$\beta \leq_a \gamma \iff \beta \in \gamma \text{ or } \beta = \gamma.$$

Clearly, $\langle \alpha, \leq_\alpha \rangle$ is an ordered set.

It is easy to verify that $\mathfrak{A}$ is isomorphic to a substructure of $\mathfrak{S}[\langle \alpha, \leq_\alpha \rangle, \mathscr{P} \mathrm{H}_P]$, and thus, $\mathfrak{A}$ is a generalized ultrametric semilattice.

If the normal logic program P of Example 3.12 is a so-called "locally hierarchical" program, then the level mapping l can be chosen so that P can be modelled as a strictly contracting function on $\mathfrak{A}$, and in that case, the fixed-point theory of [26] can be directly applied to obtain, constructively, the unique supported model of P (see [11]).

4 Representation

We want to prove that for every sentence in the first-order language of Σ, that sentence is true in every standard generalized ultrametric semilattice of linear signals if and only if it is deducible from the two sentences corresponding to the two clauses of Proposition 2.9, along, of course, with the standard axioms for semilattices and generalized ultrametric spaces with totally ordered distance sets. The "if" part will follow from Proposition 3.9 and 3.8, and the soundness theorem for first-order logic. But the "only if" part will need more work. Our purpose in this section is to prove that every generalized ultrametric semilattice with a totally ordered distance set is isomorphic to a standard generalized ultrametric semilattice of linear signals; the "only if" part will then follow from Gödel's completeness theorem.

Assume a generalized ultrametric semilattice $\mathfrak{A}$ such that $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle$ is totally ordered.

For notational convenience, we will informally write $\geq^{\mathfrak{A}}$ for the inverse of $\leq^{\mathfrak{A}}$, and $<^{\mathfrak{A}}$ and $>^{\mathfrak{A}}$ for the irreflexive parts of $\leq^{\mathfrak{A}}$ and $\geq^{\mathfrak{A}}$ respectively. Again, all this can be done formally, but once more, we shall not worry about the details.

We want to construct a standard generalized ultrametric semilattice of linear signals $\mathfrak{A}'$ that is isomorphic to $\mathfrak{A}$. The first thing we need to do is choose the tag set that we are going to use. Clearly, there is an inverse relationship between tags and distances; the smaller the tags at which two signals differ, the larger the distance between the two signals. What we might try then is use $|\mathfrak{A}|_{\mathrm{D}}$, ordered by the inverse of $\leq^{\mathfrak{A}}$, namely $\geq^{\mathfrak{A}}$. But since the least element of $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle$, namely $\mathbf{0}^{\mathfrak{A}}$, must correspond to the least element of $\langle |\mathfrak{A}'|_{\mathrm{D}}, \leq^{\mathfrak{A}'} \rangle$, which will be the chosen tag set itself, we will use $|\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\}$ instead, ordered by the restriction of $\geq^{\mathfrak{A}}$ to $|\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\}$, and let each $d \in |\mathfrak{A}|_{\mathrm{D}}$ correspond to the distance $\{d' \mid d' >^{\mathfrak{A}} d\}$ in $|\mathfrak{A}'|_{\mathrm{D}}$ (see also Proposition 4.8).

Note 4.1. Using $|\mathfrak{A}|_D$, ordered by $\geq^{\mathfrak{A}}$, and letting each $d \in |\mathfrak{A}|_D$ correspond to the distance $\{d' \mid d' \geq^{\mathfrak{A}} d\}$ in $|\mathfrak{A}'|_D$, does not work, in general.

Example 4.1.1. Let V be a set, and v_1 and v_2 two distinct members of V.

Let $x_1 = \{\langle 0, v_1 \rangle\}$.

Let $x_2 = \{\langle 0, v_2 \rangle\}$.

For every $n \in \mathbb{N}$, let $y_1(n) = \{\langle 0, v_1 \rangle, \langle \frac{1}{n+1}, v_1 \rangle\}$.

For every $n \in \mathbb{N}$, let $y_2(n) = \{\langle 0, v_2 \rangle, \langle \frac{1}{n+1}, v_2 \rangle\}$.

Let $A = \{\emptyset\} \cup \{x_1, x_2\} \cup \{y_1(n), y_2(n) \mid n \in \mathbb{N}\}$.

Let $D = \{d_{S[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V]}(s_1, s_2) \mid s_1, s_2 \in A\}$.

Then

$$D = \{\{r \mid r <_{\mathbb{Q}} 0\}\} \cup \{\{r \mid r <_{\mathbb{Q}} \tfrac{1}{n+1}\} \mid n \in \mathbb{N}\} \cup \{\mathbb{Q}\}.$$

Let $\sqcap_A$ be the restriction of $\sqcap_{S[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V]}$ to $A \times A$.

Let d_A be the restriction of $d_{S[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V]}$ to $A \times A$.

Let $\mathfrak{A}$ be a Σ-structure such that $|\mathfrak{A}|_A = A$, $|\mathfrak{A}|_D = D$, and the following are true:

1. $\sqcap^{\mathfrak{A}} = \sqcap_A$;
2. $\leq^{\mathfrak{A}} = \supseteq_D$;
3. $\mathbf{0}^{\mathfrak{A}} = \mathbb{Q}$;
4. $\mathbf{d}^{\mathfrak{A}} = d_A$.

Clearly, $\langle A, \sqcap_A \rangle$ is a subsemilattice of $\langle S[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V], \sqcap_{S[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V]} \rangle$. Thus, $\mathfrak{A}$ is a substructure of $\mathfrak{S}[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V]$, and hence, by Proposition 3.9 and 3.8, $\mathfrak{A}$ is a generalized ultrametric semilattice. And clearly, $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}} \rangle$ is totally ordered.

Suppose, toward contradiction, that there is a set V' such that there is an isomorphism $\{h_A, h_D\}$ between $\mathfrak{A}$ and a substructure $\mathfrak{A}'$ of $\mathfrak{S}[\langle |\mathfrak{A}|_D, \geq^{\mathfrak{A}} \rangle, V']$, and for every $d \in |\mathfrak{A}|_D$, $h_D(d) = \{d' \mid d' \geq^{\mathfrak{A}} d\}$. Then

$$\begin{aligned}
d^{\mathfrak{A}'}(h_A(x_1), h_A(x_2)) &= h_D(d^{\mathfrak{A}}(x_1, x_2)) \\
&= h_D(\{r \mid r <_{\mathbb{Q}} 0\}) \\
&= \{\{r \mid r <_{\mathbb{Q}} 0\}\}.
\end{aligned}$$

Thus, there is $d \in \{\{r \mid r <_{\mathbb{Q}} \frac{1}{n+1}\} \mid n \in \mathbb{N}\} \cup \{\mathbb{Q}\}$ such that $h_A(x_1)$ is defined at d, or $h_A(x_2)$ is defined at d. Without loss of generality, assume that $h_A(x_1)$ is defined at d. Then for every $n \in \mathbb{N}$,

$$d^{\mathfrak{A}'}(h_A(x_1), h_A(y_1(n))) \leq^{\mathfrak{A}'} \{d' \mid d' \geq^{\mathfrak{A}} d\},$$

and thus, by the pigeonhole principle, there is $n_1, n_2 \in \mathbb{N}$ such that

$$d^{\mathfrak{A}'}(h_A(x_1), h_A(y_1(n_1))) = d^{\mathfrak{A}'}(h_A(x_1), h_A(y_1(n_2))),$$

contrary to $\{h_A, h_D\}$ being an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}'$.

Therefore, for every set V', there is no isomorphism between $\mathfrak{A}$ and a substructure of $\mathfrak{S}[\langle |\mathfrak{A}|_D, \geq^{\mathfrak{A}} \rangle, V']$ such that for every $d \in |\mathfrak{A}|_D$, $h_D(d) = \{d' \mid d' \geq^{\mathfrak{A}} d\}$.

Now, with this set of tags, a suitable set of values is the set of all open balls[21] in $\langle |\mathfrak{A}|_A, |\mathfrak{A}|_D, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}}, \mathbf{d}^{\mathfrak{A}} \rangle$. This is because, in generalized ultrametric spaces with totally ordered distance sets, by the strict generalized ultrametric inequality, every point inside an open ball is a center of that open ball. Therefore, if for every $a \in |\mathfrak{A}|_A$ and any $d \in |\mathfrak{A}|_D \setminus \{\mathbf{0}^{\mathfrak{A}}\}$, we arrange for the signal corresponding to a to have the value $\{a' \mid \mathbf{d}^{\mathfrak{A}}(a', a) <^{\mathfrak{A}} d\}$ at d, assuming, of course, that signal is to have a value at d, we are assured that for every $a_1, a_2 \in |\mathfrak{A}|_A$, the values of the signals corresponding to a_1 and a_2 at any tag $d >^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)$, if any, will be the same, and those at every tag $d \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)$, if any, will be different. And this is consistent with our intention to let each $d \in |\mathfrak{A}|_D$ correspond to the distance $\{d' \mid d' >^{\mathfrak{A}} d\}$ in $|\mathfrak{A}'|_D$.

What remains, of course, is to decide for every $a \in |\mathfrak{A}|_A$ and any $d \in |\mathfrak{A}|_D \setminus \{\mathbf{0}^{\mathfrak{A}}\}$, whether the signal corresponding to a is to have a value at d or not. Suppose that for every $a \in |\mathfrak{A}|_A$ and any $d \in |\mathfrak{A}|_D \setminus \{\mathbf{0}^{\mathfrak{A}}\}$, the signal corresponding to a were to have a value at d, and in particular, the value $\{a' \mid \mathbf{d}^{\mathfrak{A}}(a', a) <^{\mathfrak{A}} d\}$. Then the suggested correspondence would satisfy clauses 2, 3, and 4 of Definition 3.2, as well as the additional properties of an embedding. But unless $|\mathfrak{A}|_A$ were a singleton, the range of that correspondence would not form a semilattice under the standard meet operation on signals of that kind. Indeed, no signal in that range would be a prefix of any other signal in that range. What we want to do is pluck enough values off those signals to impose the right ordering among them, but leave enough values on them to preserve the distances between them. But how are we to decide which ones to pluck and which ones to leave? The following is instrumental:

Proposition 4.1. *If $\langle T, \leq_T \rangle$ is totally ordered, then for every $s \in S[\langle T, \leq_T \rangle, V]$ and any $t \in T$, if there is s' such that*

$$d_{S[\langle T, \leq_T \rangle, V]}(s, s') \subseteq \{t' \mid t' <_T t\}$$

and

$$d_{S[\langle T, \leq_T \rangle, V]}(s, s \sqcap_{S[\langle T, \leq_T \rangle, V]} s') \supset \{t' \mid t' <_T t\},$$

then s is undefined at t.

Proof. Suppose that $\langle T, \leq_T \rangle$ is totally ordered.

Assume $s \in S[\langle T, \leq_T \rangle, V]$ and $t \in T$.

Suppose that there is s' such that

$$d_{S[\langle T, \leq_T \rangle, V]}(s, s') \subseteq \{t' \mid t' <_T t\} \tag{1}$$

and

$$d_{S[\langle T, \leq_T \rangle, V]}(s, s \sqcap_{S[\langle T, \leq_T \rangle, V]} s') \supset \{t' \mid t' <_T t\}. \tag{2}$$

By (1) and Proposition 2.8, $s \sqcap_{S[\langle T, \leq_T \rangle, V]} s'$ is undefined at t, and since $\langle T, \leq_T \rangle$ is totally ordered, by (2),

$$s(t) \simeq (s \sqcap_{S[\langle T, \leq_T \rangle, V]} s')(t).$$

Thus, s is undefined at t. $\qquad\square$

By Proposition 4.1, since we intend to let each $d \in |\mathfrak{A}|_D$ correspond to the distance $\{d' \mid d' >^{\mathfrak{A}} d\}$ in $|\mathfrak{A}'|_D$, for every $a \in |\mathfrak{A}|_A$ and any $d \in |\mathfrak{A}|_D \setminus \{\mathbf{0}^{\mathfrak{A}}\}$, if there is a' such that $\mathbf{d}^{\mathfrak{A}}(a, a') \geq^{\mathfrak{A}} d$ and $\mathbf{d}^{\mathfrak{A}}(a, a \sqcap^{\mathfrak{A}} a') <^{\mathfrak{A}} d$, the signal corresponding to a is not to have a value at d. Of course, Proposition 4.1 does not tell us which values must be left, only which ones must be plucked. But as it turns out, plucking just those is enough.

---

[21] For every generalized ultrametric space $\langle A, P, \leqslant, 0, d \rangle$, and every $B \subseteq A$, B is an *open ball* in $\langle A, P, \leqslant, 0, d \rangle$ if and only if there is $a \in A$ and $p \in P \setminus \{0\}$ such that $B = \{a' \mid d(a', a) < p\}$.

We will need to introduce some notation.

We write $\mathrm{T}[\mathfrak{A}]$ for $|\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\}$, and $\leq_{\mathrm{T}[\mathfrak{A}]}$ for the restriction of $\geq^{\mathfrak{A}}$ to $|\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\}$.

We write $\mathrm{V}[\mathfrak{A}]$ for $\mathscr{B}_{\mathrm{open}} \langle |\mathfrak{A}|_{\mathrm{A}}, |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}}, \mathbf{d}^{\mathfrak{A}} \rangle$.[22]

We write $\mathfrak{S}[\mathfrak{A}]$ for $\mathfrak{S}[\langle \mathrm{T}[\mathfrak{A}], \leq_{\mathrm{T}[\mathfrak{A}]} \rangle, \mathrm{V}[\mathfrak{A}]]$.

We write $\mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}$ for a function from $|\mathfrak{A}|_{\mathrm{A}}$ to $|\mathfrak{S}[\mathfrak{A}]|_{\mathrm{A}}$ such that for every $a \in |\mathfrak{A}|_{\mathrm{A}}$ and every $d \in \mathrm{T}[\mathfrak{A}]$,

$$\mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}(a)(d) \simeq \begin{cases} \text{undefined} & \text{if there is } a' \text{ such that } \mathbf{d}^{\mathfrak{A}}(a, a') \geq^{\mathfrak{A}} d \text{ and } \mathbf{d}^{\mathfrak{A}}(a, a \sqcap^{\mathfrak{A}} a') <^{\mathfrak{A}} d; \\ \{a' \mid \mathbf{d}^{\mathfrak{A}}(a', a) <^{\mathfrak{A}} d\} & \text{otherwise.} \end{cases}$$

We write $\mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}$ for a function from $|\mathfrak{A}|_{\mathrm{D}}$ to $|\mathfrak{S}[\mathfrak{A}]|_{\mathrm{D}}$ such that for every $d \in |\mathfrak{A}|_{\mathrm{D}}$,

$$\mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}(d) = \{d' \mid d' >^{\mathfrak{A}} d\}.$$

The following is immediate:

Proposition 4.2. *The following are true:*

1. *for every $d_1, d_2 \in |\mathfrak{A}|_{\mathrm{D}}$, $d_1 \leq^{\mathfrak{A}} d_2$ if and only if $\mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}(d_1) \leq^{\mathfrak{S}[\mathfrak{A}]} \mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}(d_2)$;*
2. *$\mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}(\mathbf{0}^{\mathfrak{A}}) = \mathbf{0}^{\mathfrak{S}[\mathfrak{A}]}$.*

By Proposition 4.2, $\mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}$ is an order-embedding of $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle$ into $\langle |\mathfrak{S}[\mathfrak{A}]|_{\mathrm{D}}, \leq^{\mathfrak{S}[\mathfrak{A}]} \rangle$ that "carries" the least element of $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle$ into the least element of $\langle |\mathfrak{S}[\mathfrak{A}]|_{\mathrm{D}}, \leq^{\mathfrak{S}[\mathfrak{A}]} \rangle$. It remains to prove that $\mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}$ is an embedding of $\langle |\mathfrak{A}|_{\mathrm{A}}, \sqcap^{\mathfrak{A}} \rangle$ into $\langle |\mathfrak{S}[\mathfrak{A}]|_{\mathrm{A}}, \sqcap^{\mathfrak{S}[\mathfrak{A}]} \rangle$, and that $\{\mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}, \mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}\}$ "preserves" distances in the sense of Definition 3.2.4. We start with the latter.

Proposition 4.3. *For every $a_1, a_2 \in |\mathfrak{A}|_{\mathrm{A}}$, $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \leq^{\mathfrak{A}} d$ if and only if $\mathbf{d}^{\mathfrak{S}[\mathfrak{A}]}(\mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}(a_1), \mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}(a_2)) \supseteq \mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}(d)$.*

Proof. Assume $a_1, a_2 \in |\mathfrak{A}|_{\mathrm{A}}$.

Suppose that

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \leq^{\mathfrak{A}} d. \tag{3}$$

If $d = \mathbf{0}^{\mathfrak{A}}$, then

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) = \mathbf{0}^{\mathfrak{A}},$$

and thus, $a_1 = a_2$. Thus,

$$\begin{aligned} \mathbf{d}^{\mathfrak{S}[\mathfrak{A}]}(\mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}(a_1), \mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}(a_2)) &= \mathbf{d}^{\mathfrak{S}[\mathfrak{A}]}(\mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}(a_1), \mathrm{h}^{\mathfrak{A}}_{\mathrm{A}}(a_1)) \\ &= |\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\} \\ &= \mathrm{h}^{\mathfrak{A}}_{\mathrm{D}}(\mathbf{0}^{\mathfrak{A}}). \end{aligned}$$

Otherwise, $d \in |\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\}$.

Assume $d' \in |\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\}$ such that $d' >^{\mathfrak{A}} d$.

---

[22] For every generalized ultrametric space $\langle A, P, \leqslant, 0, d \rangle$, we write $\mathscr{B}_{\mathrm{open}} \langle A, P, \leqslant, 0, d \rangle$ for the set of all open balls in $\langle A, P, \leqslant, 0, d \rangle$.

Then, by (3),

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) <^{\mathfrak{A}} d'. \tag{4}$$

If $d' \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)$ and $d' \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)$, then, by (4) and the strict generalized ultrametric inequality,

$$
\begin{aligned}
\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)(d') &= \{a \mid \mathbf{d}^{\mathfrak{A}}(a, a_1) <^{\mathfrak{A}} d'\} \\
&= \{a \mid \mathbf{d}^{\mathfrak{A}}(a, a_2) <^{\mathfrak{A}} d'\} \\
&= \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)(d').
\end{aligned}
$$

Otherwise, $d' \notin \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)$ or $d' \notin \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)$. Without loss of generality, assume that $d' \notin \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)$. Then there is a such that

$$\mathbf{d}^{\mathfrak{A}}(a_1, a) \geq^{\mathfrak{A}} d' \tag{5}$$

and

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_1 \sqcap^{\mathfrak{A}} a) <^{\mathfrak{A}} d'. \tag{6}$$

Suppose, toward contradiction, that

$$\mathbf{d}^{\mathfrak{A}}(a_2, a) <^{\mathfrak{A}} d'.$$

Then, by (4) and the strict generalized ultrametric inequality,

$$\mathbf{d}^{\mathfrak{A}}(a_1, a) <^{\mathfrak{A}} d',$$

contrary to (5).
Therefore,

$$\mathbf{d}^{\mathfrak{A}}(a_2, a) \geq^{\mathfrak{A}} d'. \tag{7}$$

By Definition 3.7.4b,

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a, a_2 \sqcap^{\mathfrak{A}} a) \leq \mathbf{d}^{\mathfrak{A}}(a_1, a_2),$$

and thus, by (4),

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a, a_2 \sqcap^{\mathfrak{A}} a) <^{\mathfrak{A}} d'. \tag{8}$$

By (4), (6), and the strict generalized ultrametric inequality,

$$\mathbf{d}^{\mathfrak{A}}(a_2, a_1 \sqcap^{\mathfrak{A}} a) <^{\mathfrak{A}} d',$$

and thus, by (8) and the strict generalized ultrametric inequality,

$$\mathbf{d}^{\mathfrak{A}}(a_2, a_2 \sqcap^{\mathfrak{A}} a) <^{\mathfrak{A}} d'. \tag{9}$$

By (7) and (9), $d' \notin \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)$, and thus,

$$\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)(d') \simeq \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)(d').$$

Thus, by generalization,

$$
\begin{aligned}
\mathbf{d}^{\mathfrak{G}[\mathfrak{A}]}(\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1), \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)) &\supseteq \{d' \mid d' >^{\mathfrak{A}} d\} \\
&= \mathrm{h}_{\mathrm{D}}^{\mathfrak{A}}(d).
\end{aligned}
$$

Conversely, suppose that

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \not\leq^{\mathfrak{A}} d.$$

Then

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) >^{\mathfrak{A}} d, \tag{10}$$

and thus, $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \in |\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\}$.

Suppose, toward contradiction, that $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \notin \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)$ and $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \notin \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)$. Then there is a_1' such that

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_1') \geq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2) \tag{11}$$

and

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_1 \sqcap^{\mathfrak{A}} a_1') <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2), \tag{12}$$

and a_2' such that

$$\mathbf{d}^{\mathfrak{A}}(a_2, a_2') \geq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2) \tag{13}$$

and

$$\mathbf{d}^{\mathfrak{A}}(a_2, a_2 \sqcap^{\mathfrak{A}} a_2') <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2). \tag{14}$$

By (11) and Definition 3.7.4a,

$$a_1 \sqcap^{\mathfrak{A}} a_1' \sqsubseteq^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_2, \tag{15}$$

and by (13) and Definition 3.7.4a,

$$a_2 \sqcap^{\mathfrak{A}} a_2' \sqsubseteq^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_2. \tag{16}$$

By (12) and Definition 3.7.4a,

$$\begin{aligned}
a_1 \sqcap^{\mathfrak{A}} a_2 &\sqsubseteq^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_1' \\
&= a_1 \sqcap^{\mathfrak{A}} a_1',
\end{aligned} \tag{17}$$

and by (14) and Definition 3.7.4a,

$$\begin{aligned}
a_1 \sqcap^{\mathfrak{A}} a_2 &\sqsubseteq^{\mathfrak{A}} a_2 \sqcap^{\mathfrak{A}} a_2 \sqcap^{\mathfrak{A}} a_2' \\
&= a_2 \sqcap^{\mathfrak{A}} a_2'.
\end{aligned} \tag{18}$$

By (15), (16), (17), and (18),

$$\begin{aligned}
a_1 \sqcap^{\mathfrak{A}} a_1' &= a_1 \sqcap^{\mathfrak{A}} a_2 \\
&= a_2 \sqcap^{\mathfrak{A}} a_2',
\end{aligned}$$

and thus, by (12), (14), and the strict generalized ultrametric inequality,

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2),$$

obtaining a contradiction.

Therefore, $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)$ or $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)$. Without loss of generality, assume that $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)$. Then

$$\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1) = \{a \mid \mathbf{d}^{\mathfrak{A}}(a, a_1) <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\},$$

and thus,

$$\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)(\mathbf{d}^{\mathfrak{A}}(a_1, a_2)) \not\simeq \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)(\mathbf{d}^{\mathfrak{A}}(a_1, a_2)).$$

Thus, by (10),

$$\mathbf{d}^{\mathfrak{G}[\mathfrak{A}]}(\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1), \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)) \not\supseteq \{d' \mid d' >^{\mathfrak{A}} d\}$$
$$= \mathrm{h}_{\mathrm{D}}^{\mathfrak{A}}(d). \qquad \square$$

The following is immediate from Proposition 4.2 and 4.3:

Proposition 4.4. $\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}$ *is one-to-one.*

Proposition 4.5. *For every* $a_1, a_2 \in |\mathfrak{A}|_{\mathrm{A}}$, $\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2) = \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1) \sqcap^{\mathfrak{G}[\mathfrak{A}]} \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_2)$.

Proof. Assume $a_1, a_2 \in |\mathfrak{A}|_{\mathrm{A}}$.
Assume $d \in |\mathfrak{A}|_{\mathrm{D}} \setminus \{\mathbf{0}^{\mathfrak{A}}\}$.
If

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) <^{\mathfrak{A}} d,$$

then, by Definition 3.7.4b,

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1) <^{\mathfrak{A}} d,$$

and thus, by Proposition 4.3,

$$\mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2)(d) \simeq \mathrm{h}_{\mathrm{A}}^{\mathfrak{A}}(a_1)(d).$$

Otherwise,

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \geq^{\mathfrak{A}} d. \tag{19}$$

By Definition 3.7.4b and the strict generalized ultrametric inequality,

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1) = \mathbf{d}^{\mathfrak{A}}(a_1, a_2)$$

or

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_2) = \mathbf{d}^{\mathfrak{A}}(a_1, a_2).$$

Without loss of generality, assume that

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1) = \mathbf{d}^{\mathfrak{A}}(a_1, a_2).$$

Then, by (19),

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1) \geq^{\mathfrak{A}} d. \tag{20}$$

However,

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, (a_1 \sqcap^{\mathfrak{A}} a_2) \sqcap^{\mathfrak{A}} a_1) = \mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1 \sqcap^{\mathfrak{A}} a_2)$$
$$= \mathbf{0}^{\mathfrak{A}}$$
$$<^{\mathfrak{A}} d. \tag{21}$$

Thus, by (20) and (21), $d \notin \mathsf{dom}\, \mathrm{h}_A^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2)$.

By generalization and extensionality, and Proposition 4.3,

$$\mathrm{h}_A^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2) = \mathrm{h}_A^{\mathfrak{A}}(a_1) \upharpoonright \{d \mid d >^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}$$
$$= \mathrm{h}_A^{\mathfrak{A}}(a_1) \upharpoonright \mathrm{h}_D^{\mathfrak{A}}(\mathbf{d}^{\mathfrak{A}}(a_1, a_2))$$
$$= \mathrm{h}_A^{\mathfrak{A}}(a_1) \upharpoonright \mathbf{d}^{\mathfrak{S}[\mathfrak{A}]}(\mathrm{h}_A^{\mathfrak{A}}(a_1), \mathrm{h}_A^{\mathfrak{A}}(a_2)),$$

and thus, by Proposition 2.8,

$$\mathrm{h}_A^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2) = \mathrm{h}_A^{\mathfrak{A}}(a_1) \sqcap^{\mathfrak{S}[\mathfrak{A}]} \mathrm{h}_A^{\mathfrak{A}}(a_2). \qquad \square$$

The following is immediate from Proposition 4.2, 4.3, 4.4, and 4.5:

Theorem 4.6. $\{\mathrm{h}_A^{\mathfrak{A}}, \mathrm{h}_D^{\mathfrak{A}}\}$ *is an embedding of* $\mathfrak{A}$ *into* $\mathfrak{S}[\mathfrak{A}]$.

And the following is immediate from Theorem 4.6 and Proposition 3.5:

Theorem 4.7. *Every generalized ultrametric semilattice with a totally ordered distance set is isomorphic to a standard generalized ultrametric semilattice of linear signals.*

Note 4.2. There is another way to construct a standard generalized ultrametric semilattice of linear signals $\mathfrak{A}'$ that is isomorphic to $\mathfrak{A}$ that involves a different choice of tag set. The idea is to use the set of all ideals[23] of $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}} \rangle$, ordered by inverse inclusion, and let each $d \in |\mathfrak{A}|_D$ correspond to the distance $\{I \mid I$ is an ideal of $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}} \rangle$, and $I \supseteq \{d' \mid d' \leq^{\mathfrak{A}} d\}\}$ in $|\mathfrak{A}'|_D$. The use of ideals is motivated by Example 4.1.1.

With this set of tags, a suitable set of values is the set of all sets of the form $\{a' \mid \mathbf{d}^{\mathfrak{A}}(a', a) \in I\}$, where $a \in |\mathfrak{A}|_A$, and I is an ideal of $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}} \rangle$. This is justified by the next proposition.

Assume a generalized ultrametric space $\langle A, P, \leqslant, 0, d \rangle$.

Proposition 4.2.1. *For every ideal I of $\langle P, \leqslant \rangle$, if $d(x, y) \in I$ and $d(y, z) \in I$, then $d(x, z) \in I$.*

Proof. Assume an ideal I of $\langle P, \leqslant \rangle$.

Suppose that $d(x, y) \in I$ and $d(y, z) \in I$.

Since I is an ideal of $\langle P, \leqslant \rangle$, there is $p \in I$ such that $d(x, y) \leq p$ and $d(y, z) \leq p$. Thus, by the generalized ultrametric inequality, $d(x, z) \leq p$, and hence, since I is an ideal of $\langle P, \leqslant \rangle$, $d(x, z) \in I$. $\qquad \square$

Just like Proposition 4.1 before, the following tells us which values must be plucked:

Proposition 4.2.2. *If $\langle T, \leq_T \rangle$ is totally ordered, for every $s \in \mathrm{S}[\langle T, \leq_T \rangle, V]$ and any $t \in T$, if there is s' such that*

$$\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s, s') \subset \{t' \mid t' \leq_T t\}$$

and

$$\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s, s \sqcap_{\mathrm{S}[\langle T, \leq_T \rangle, V]} s') \supseteq \{t' \mid t' \leq_T t\},$$

then s is undefined at t.

Proof. Suppose that $\langle T, \leq_T \rangle$ is totally ordered.

Assume $s \in \mathrm{S}[\langle T, \leq_T \rangle, V]$ and $t \in T$.

Suppose that there is s' such that

$$\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s, s') \subset \{t' \mid t' \leq_T t\} \tag{22}$$

and

$$\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s, s \sqcap_{\mathrm{S}[\langle T, \leq_T \rangle, V]} s') \supseteq \{t' \mid t' \leq_T t\}. \tag{23}$$

Since $\langle T, \leq_T \rangle$ is totally ordered, by (22) and Proposition 2.8, $s \sqcap_{\mathrm{S}[\langle T, \leq_T \rangle, V]} s'$ is undefined at t, and by (23),

$$s(t) \simeq (s \sqcap_{\mathrm{S}[\langle T, \leq_T \rangle, V]} s')(t).$$

Thus, s is undefined at t. $\qquad\square$

Again, we will need to introduce some notation.

We write $\mathrm{T}'[\mathfrak{A}]$ for $\{I \mid I \text{ is an ideal of } \langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle\}$, and $\leq_{\mathrm{T}'[\mathfrak{A}]}$ for $\supseteq_{\{I \mid I \text{ is an ideal of } \langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle\}}$.

We write $\mathrm{V}'[\mathfrak{A}]$ for $\{\{a' \mid \mathbf{d}^{\mathfrak{A}}(a', a) \in I\} \mid a \in |\mathfrak{A}|_{\mathrm{A}} \text{ and } I \text{ is an ideal of } \langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle\}$.

We write $\mathfrak{S}'[\mathfrak{A}]$ for $\mathfrak{S}[\langle \mathrm{T}'[\mathfrak{A}], \leq_{\mathrm{T}'[\mathfrak{A}]} \rangle, \mathrm{V}'[\mathfrak{A}]]$.

We write $\mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}$ for a function from $|\mathfrak{A}|_{\mathrm{A}}$ to $|\mathfrak{S}'[\mathfrak{A}]|_{\mathrm{A}}$ such that for every $a \in |\mathfrak{A}|_{\mathrm{A}}$ and every ideal I of $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle$,

$$\mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a)(I) \simeq \begin{cases} \text{undefined} & \text{if there is } a' \text{ such that } \mathbf{d}^{\mathfrak{A}}(a, a') \notin I \text{ and } \mathbf{d}^{\mathfrak{A}}(a, a \sqcap^{\mathfrak{A}} a') \in I; \\ \{a' \mid \mathbf{d}^{\mathfrak{A}}(a', a) \in I\} & \text{otherwise.} \end{cases}$$

We write $\mathrm{h}'^{\mathfrak{A}}_{\mathrm{D}}$ for a function from $|\mathfrak{A}|_{\mathrm{D}}$ to $|\mathfrak{S}'[\mathfrak{A}]|_{\mathrm{D}}$ such that for every $d \in |\mathfrak{A}|_{\mathrm{D}}$,

$$\mathrm{h}'^{\mathfrak{A}}_{\mathrm{D}}(d) = \{I \mid I \text{ is an ideal of } \langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle, \text{ and } I \supseteq \{d' \mid d' \leq^{\mathfrak{A}} d\}\}.$$

The following is immediate:

Proposition 4.2.3. *The following are true:*

 1. *for every $d_1, d_2 \in |\mathfrak{A}|_{\mathrm{D}}$, $d_1 \leq^{\mathfrak{A}} d_2$ if and only if $\mathrm{h}'^{\mathfrak{A}}_{\mathrm{D}}(d_1) \leq^{\mathfrak{S}'[\mathfrak{A}]} \mathrm{h}'^{\mathfrak{A}}_{\mathrm{D}}(d_2)$;*
 2. *$\mathrm{h}'^{\mathfrak{A}}_{\mathrm{D}}(\mathbf{0}^{\mathfrak{A}}) = \mathbf{0}^{\mathfrak{S}'[\mathfrak{A}]}$.*

Proposition 4.2.4. *For every $a_1, a_2 \in |\mathfrak{A}|_{\mathrm{A}}$, $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \leq^{\mathfrak{A}} d$ if and only if $\mathbf{d}^{\mathfrak{S}'[\mathfrak{A}]}(\mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_1), \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_2)) \supseteq \mathrm{h}'^{\mathfrak{A}}_{\mathrm{D}}(d)$.*

Proof. Assume $a_1, a_2 \in |\mathfrak{A}|_{\mathrm{A}}$.

Suppose that

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \leq^{\mathfrak{A}} d. \tag{24}$$

Assume an ideal I of $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle$ such that $I \supseteq \{ d' \mid d' \leq^{\mathfrak{A}} d \}$.

By (24),

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \in I. \tag{25}$$

If $I \in \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_1)$ and $I \in \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_2)$, then, by (25) and Proposition 4.2.1,

$$\begin{aligned}
\mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_1)(I) &= \{ a \mid \mathbf{d}^{\mathfrak{A}}(a, a_1) \in I \} \\
&= \{ a \mid \mathbf{d}^{\mathfrak{A}}(a, a_2) \in I \} \\
&= \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_2)(I).
\end{aligned}$$

Otherwise, $I \notin \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_1)$ or $I \notin \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_2)$. Without loss of generality, assume that $I \notin \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_1)$. Then there is a such that

$$\mathbf{d}^{\mathfrak{A}}(a_1, a) \notin I \tag{26}$$

and

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_1 \sqcap^{\mathfrak{A}} a) \in I. \tag{27}$$

Suppose, toward contradiction, that

$$\mathbf{d}^{\mathfrak{A}}(a_2, a) \in I.$$

Then, by (25) and Proposition 4.2.1,

$$\mathbf{d}^{\mathfrak{A}}(a_1, a) \in I.$$

contrary to (26).

Therefore,

$$\mathbf{d}^{\mathfrak{A}}(a_2, a) \notin I. \tag{28}$$

By Definition 3.7.4b,

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a, a_2 \sqcap^{\mathfrak{A}} a) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2),$$

and thus, since I is an ideal of $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle$, by (25),

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a, a_2 \sqcap^{\mathfrak{A}} a) \in I. \tag{29}$$

Also, by (25), (27), and Proposition 4.2.1,

$$\mathbf{d}^{\mathfrak{A}}(a_2, a_1 \sqcap^{\mathfrak{A}} a) \in I,$$

and thus, by (29) and Proposition 4.2.1,

$$\mathbf{d}^{\mathfrak{A}}(a_2, a_2 \sqcap^{\mathfrak{A}} a) \in I. \tag{30}$$

By (28) and (30), $I \notin \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_2)$, and thus,

$$\mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_1)(I) \simeq \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_2)(I).$$

Thus, by generalization,

$$\mathbf{d}^{\mathfrak{S}'[\mathfrak{A}]}(\mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_1), \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_2)) \supseteq \{I \mid I \text{ is an ideal of } \langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle, \text{ and } I \supseteq \{d' \mid d' \leq^{\mathfrak{A}} d\}\}$$
$$= \mathrm{h}'^{\mathfrak{A}}_{\mathrm{D}}(d).$$

Conversely, suppose that

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \not\leq^{\mathfrak{A}} d.$$

Then

$$\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \supseteq \{d' \mid d' \leq^{\mathfrak{A}} d\}. \tag{31}$$

Suppose, toward contradiction, that $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \notin \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_1)$ and $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \notin \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_{\mathrm{A}}(a_2)$. Then there is a'_1 such that

$$\mathbf{d}^{\mathfrak{A}}(a_1, a'_1) \notin \{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \tag{32}$$

and

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_1 \sqcap^{\mathfrak{A}} a'_1) \in \{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}, \tag{33}$$

and a'_2 such that

$$\mathbf{d}^{\mathfrak{A}}(a_2, a'_2) \notin \{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \tag{34}$$

and

$$\mathbf{d}^{\mathfrak{A}}(a_2, a_2 \sqcap^{\mathfrak{A}} a'_2) \in \{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}. \tag{35}$$

By (32) and (33),

$$\mathbf{d}^{\mathfrak{A}}(a_1, a'_1) \geq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2) \tag{36}$$

and

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_1 \sqcap^{\mathfrak{A}} a'_1) <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2), \tag{37}$$

and by (34) and (35),

$$\mathbf{d}^{\mathfrak{A}}(a_2, a'_2) \geq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2) \tag{38}$$

and

$$\mathbf{d}^{\mathfrak{A}}(a_2, a_2 \sqcap^{\mathfrak{A}} a'_2) <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2). \tag{39}$$

By (36) and Definition 3.7.4a,

$$a_1 \sqcap^{\mathfrak{A}} a'_1 \sqsubseteq^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_2, \tag{40}$$

and by (37) and Definition 3.7.4a,

$$a_2 \sqcap^{\mathfrak{A}} a_2' \sqsubseteq^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_2. \tag{41}$$

By (38) and Definition 3.7.4a,

$$a_1 \sqcap^{\mathfrak{A}} a_2 \sqsubseteq^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_1 \sqcap^{\mathfrak{A}} a_1'$$
$$= a_1 \sqcap^{\mathfrak{A}} a_1', \tag{42}$$

and by (39) and Definition 3.7.4a,

$$a_1 \sqcap^{\mathfrak{A}} a_2 \sqsubseteq^{\mathfrak{A}} a_2 \sqcap^{\mathfrak{A}} a_2 \sqcap^{\mathfrak{A}} a_2'$$
$$= a_2 \sqcap^{\mathfrak{A}} a_2'. \tag{43}$$

By (40), (41), (42), and (43),

$$a_1 \sqcap^{\mathfrak{A}} a_1' = a_1 \sqcap^{\mathfrak{A}} a_2$$
$$= a_2 \sqcap^{\mathfrak{A}} a_2',$$

and thus, by (37), (39), and the strict generalized ultrametric inequality,

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2),$$

obtaining a contradiction.

Therefore, $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1)$ or $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)$.

If $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1)$ and $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \notin \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)$, or $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \notin \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1)$ and $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)$, then

$$\mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1)(\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}) \not\simeq \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)(\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}).$$

Otherwise, $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1)$ and $\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\} \in \mathsf{dom}\, \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)$. Then

$$\mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1)(\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}) = \{a \mid \mathbf{d}^{\mathfrak{A}}(a_1, a) \in \{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}\}$$

and

$$\mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)(\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}) = \{a \mid \mathbf{d}^{\mathfrak{A}}(a_2, a) \in \{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}\}.$$

Since $0 \in \{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}$, $a_1 \in \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1)(\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\})$. However, $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \notin \{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}$, and thus, $a_1 \notin \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)(\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\})$. Thus,

$$\mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1)(\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}) \not\simeq \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)(\{d' \mid d' <^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}).$$

Thus, by (31),

$$\mathbf{d}^{\mathfrak{S}'[\mathfrak{A}]}(\mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_1), \mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}(a_2)) \not\supseteq \{I \mid I \text{ is an ideal of } \langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle, \text{ and } I \supseteq \{d' \mid d' \leq^{\mathfrak{A}} d\}\}$$
$$= \mathrm{h}_{\mathrm{D}}'^{\mathfrak{A}}(d). \qquad \square$$

The following is immediate from Proposition 4.2.3 and 4.2.4:

Proposition 4.2.5. $\mathrm{h}_{\mathrm{A}}'^{\mathfrak{A}}$ *is one-to-one.*

Proposition 4.2.6. *For every $a_1, a_2 \in |\mathfrak{A}|_A$, $\mathrm{h}'^{\mathfrak{A}}_A(a_1 \sqcap^{\mathfrak{A}} a_2) = \mathrm{h}'^{\mathfrak{A}}_A(a_1) \sqcap^{\mathfrak{S}'[\mathfrak{A}]} \mathrm{h}'^{\mathfrak{A}}_A(a_2)$.*

Proof. Assume $a_1, a_2 \in |\mathfrak{A}|_A$.

Assume an ideal I of $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}} \rangle$.

If

$$I \supseteq \{d' \mid d' \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\},$$

then, by Definition 3.7.4b,

$$I \supseteq \{d' \mid d' \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1)\},$$

and thus, by Proposition 4.2.4,

$$\mathrm{h}'^{\mathfrak{A}}_A(a_1 \sqcap^{\mathfrak{A}} a_2)(I) \simeq \mathrm{h}'^{\mathfrak{A}}_A(a_1)(I).$$

Otherwise,

$$I \not\supseteq \{d' \mid d' \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}. \tag{44}$$

Then

$$\mathbf{d}^{\mathfrak{A}}(a_1, a_2) \notin I. \tag{45}$$

By Definition 3.7.4b and the strict generalized ultrametric inequality,

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1) = \mathbf{d}^{\mathfrak{A}}(a_1, a_2)$$

or

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_2) = \mathbf{d}^{\mathfrak{A}}(a_1, a_2).$$

Without loss of generality, assume that

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1) = \mathbf{d}^{\mathfrak{A}}(a_1, a_2).$$

Then, by (46),

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1) \notin I. \tag{46}$$

However,

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, (a_1 \sqcap^{\mathfrak{A}} a_2) \sqcap^{\mathfrak{A}} a_1) = \mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, a_1 \sqcap^{\mathfrak{A}} a_2)$$
$$= \mathbf{0}^{\mathfrak{A}},$$

and thus, since I is an ideal of $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}} \rangle$,

$$\mathbf{d}^{\mathfrak{A}}(a_1 \sqcap^{\mathfrak{A}} a_2, (a_1 \sqcap^{\mathfrak{A}} a_2) \sqcap^{\mathfrak{A}} a_1) \in I. \tag{47}$$

Thus, by (46) and (47), $I \notin \mathsf{dom}\, \mathrm{h}'^{\mathfrak{A}}_A(a_1 \sqcap^{\mathfrak{A}} a_2)$.

By generalization and extensionality, and Proposition 4.2.4,

$$\mathrm{h}'^{\mathfrak{A}}_A(a_1 \sqcap^{\mathfrak{A}} a_2) = \mathrm{h}'^{\mathfrak{A}}_A(a_1) \upharpoonright \{I \mid I \text{ is an ideal of } \langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}} \rangle, \text{ and } I \supseteq \{d' \mid d' \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2)\}\}$$
$$= \mathrm{h}'^{\mathfrak{A}}_A(a_1) \upharpoonright \mathrm{h}'^{\mathfrak{A}}_D(\mathbf{d}^{\mathfrak{A}}(a_1, a_2))$$
$$= \mathrm{h}'^{\mathfrak{A}}_A(a_1) \upharpoonright \mathbf{d}^{\mathfrak{S}'[\mathfrak{A}]}(\mathrm{h}'^{\mathfrak{A}}_A(a_1), \mathrm{h}'^{\mathfrak{A}}_A(a_2)),$$

and thus, by Proposition 2.8,

$$\mathrm{h}'^{\mathfrak{A}}_A(a_1 \sqcap^{\mathfrak{A}} a_2) = \mathrm{h}'^{\mathfrak{A}}_A(a_1) \sqcap^{\mathfrak{S}'[\mathfrak{A}]} \mathrm{h}'^{\mathfrak{A}}_A(a_2). \qquad \square$$

Theorem 4.2.7. $\{h'^{\mathfrak{A}}_A, h'^{\mathfrak{A}}_D\}$ *is an embedding of* $\mathfrak{A}$ *into* $\mathfrak{S}'[\mathfrak{A}]$.

Theorem 4.7 is our prized result. It is a so-called "representation theorem", attesting that every abstract structure can be represented as a concrete one. It is, in this sense, akin, for example, to Cayley's theorem in group theory, or Stone's representation theorem for Boolean algebras. And yet, one might argue that there is a sense in which Theorem 4.7 is not quite as practical as those theorems. This is to do with the distinguished status of generalized distances that we have alluded to in Section 3. One is interested in collections of linear signals that form subsemilattices of the corresponding semilattices of all signals, but no matter the structure of any such collection, one always knows that the ordered set of distances associated with the corresponding standard generalized ultrametric space of all signals is of one particular form: it is the set of all lower sets of some totally ordered set ordered by inverse inclusion. And one might think that this extra piece of information might be used to infer additional formal properties of the relationship between the meet operation and the generalized distance function in subsemilattices of linear signals. For this reason, one might want to constrain generalized ultrametric semilattices with totally ordered distance sets in a way that reflects that extra piece of information, and prove that every such constrained generalized ultrametric semilattice is isomorphic not just to a structure in $\mathsf{S}(\mathcal{S}_{\mathrm{lin}})$, but to a structure in $\mathsf{S}_A(\mathcal{S}_{\mathrm{lin}})$.

If the distance set of a generalized ultrametric semilattice with a totally ordered distance set is finite, then that generalized ultrametric semilattice is already isomorphic to a structure in $\mathsf{S}_A(\mathcal{S}_{\mathrm{lin}})$.

Proposition 4.8. *If* $|\mathfrak{A}|_{\mathrm{D}}$ *is finite, then* $h^{\mathfrak{A}}_{\mathrm{D}}$ *is a one-to-one correspondence between* $|\mathfrak{A}|_{\mathrm{D}}$ *and* $|\mathfrak{S}[\mathfrak{A}]|_{\mathrm{D}}$.

If the distance set of a generalized ultrametric semilattice with a totally ordered distance set is infinite, then that generalized ultrametric semilattice need not be isomorphic to a structure in $\mathsf{S}_A(\mathcal{S}_{\mathrm{lin}})$. For an infinite totally ordered distance set need not be order-isomorphic to the set of all lower sets of some totally ordered set ordered by inverse inclusion. What might be surprising is that, even if the ordered set of distances of a generalized ultrametric semilattice with a totally ordered distance set is order-isomorphic to the set of all lower sets of some totally ordered set ordered by inverse inclusion, that generalized ultrametric semilattice still need not be isomorphic to a structure in $\mathsf{S}_A(\mathcal{S}_{\mathrm{lin}})$.

Example 4.9. Let V be a set, and v a member of V.

Let $A = \{\emptyset\} \cup \{\{\langle -\frac{1}{n+1}, v \rangle\} \mid n \in \mathbb{N}\} \cup \{\{\langle 0, v \rangle\}\} \cup \{\{\langle 0, v \rangle, \langle \frac{1}{n+1}, v \rangle\} \mid n \in \mathbb{N}\}$.

Let $D = \{\mathrm{d}_{\mathsf{S}[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V]}(s_1, s_2) \mid s_1, s_2 \in A\}$.

Then

$$D = \{\{r \mid r <_{\mathbb{Q}} -\tfrac{1}{n+1}\} \mid n \in \mathbb{N}\} \cup \{\{r \mid r <_{\mathbb{Q}} 0\}\} \cup \{\{r \mid r <_{\mathbb{Q}} \tfrac{1}{n+1}\} \mid n \in \mathbb{N}\} \cup \{\mathbb{Q}\}.$$

Let $\sqcap_A$ be the restriction of $\sqcap_{\mathsf{S}[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V]}$ to $A \times A$.

Let d_A be the restriction of $\mathrm{d}_{\mathsf{S}[\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle, V]}$ to $A \times A$.

Let $\mathfrak{A}$ be a Σ-structure such that $|\mathfrak{A}|_A = A$, $|\mathfrak{A}|_{\mathrm{D}} = D$, and the following are true:

1. $\sqcap^{\mathfrak{A}} = \sqcap_A$;
2. $\leq^{\mathfrak{A}} = \supseteq_D$;
3. $\mathbf{0}^{\mathfrak{A}} = \mathbb{Q}$;
4. $\mathbf{d}^{\mathfrak{A}} = \mathrm{d}_A$.

---

[23] For every ordered set $\langle P, \leqslant \rangle$, and every $I \subseteq P$, I is an *ideal* of $\langle P, \leqslant \rangle$ if and only if I is a non-empty lower set of $\langle P, \leqslant \rangle$, and for any $p_1, p_2 \in I$, there is $p \in I$ such that $p_1 \leqslant p$ and $p_2 \leqslant p$.

Clearly, $\langle A, \sqcap_A \rangle$ is a subsemilattice of $\langle \mathrm{S}[\langle \mathbb{Q}, \leq_{\mathbb{Q}}\rangle, V], \sqcap_{\mathrm{S}[\langle \mathbb{Q}, \leq_{\mathbb{Q}}\rangle, V]}\rangle$. Consequently, $\mathfrak{A}$ is a substructure of $\mathfrak{S}[\langle \mathbb{Q}, \leq_{\mathbb{Q}}\rangle, V]$, and thus, by Proposition 3.9 and 3.8, $\mathfrak{A}$ is a generalized ultrametric semilattice. And clearly, $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}} \rangle$ is totally ordered.

Let $J = \{d \mid d$ is completely join-irreducible in $\langle D, \subseteq_D \rangle\}$.[24]

Then

$$J = \{\{r \mid r <_{\mathbb{Q}} -\tfrac{1}{n+1}\} \mid n \in \mathbb{N} \setminus \{0\}\} \cup \{\{r \mid r <_{\mathbb{Q}} \tfrac{1}{n+1}\} \mid n \in \mathbb{N}\} \cup \{\mathbb{Q}\},$$

and clearly, $\langle D, \supseteq_D \rangle$ is order-isomorphic to $\langle \mathscr{L}\langle J, \subseteq_J\rangle, \supseteq_{\mathscr{L}\langle J, \subseteq_J\rangle}\rangle$.

Suppose, toward contradiction, that there is a totally ordered set $\langle T, \leq_T \rangle$ and a set V' such that there is an isomorphism $\{h_{\mathrm{A}}, h_{\mathrm{D}}\}$ between $\mathfrak{A}$ and an A-substructure of $\mathfrak{S}[\langle T, \leq_T\rangle, V']$. Then h_{D} is an order-isomorphism between $\langle D, \supseteq_D \rangle$ and $\langle \mathscr{L}\langle T, \leq_T\rangle, \supseteq_{\mathscr{L}\langle T, \leq_T\rangle}\rangle$, and thus, since $\langle D, \supseteq_D \rangle$ is order-isomorphic to $\langle \mathscr{L}\langle J, \subseteq_J\rangle, \supseteq_{\mathscr{L}\langle J, \subseteq_J\rangle}\rangle$, $\langle T, \leq_T\rangle$ and $\langle J, \subseteq_J\rangle$ are order-isomorphic (e.g., see [4, thm. 10.29]). Without loss of generality, assume that $\langle T, \leq_T\rangle = \langle J, \subseteq_J\rangle$. Since

$$
\begin{aligned}
\mathrm{d}_{\mathrm{S}[\langle T, \leq_T\rangle, V']}(h_{\mathrm{A}}(\emptyset), h_{\mathrm{A}}(\{\langle 0, v\rangle\})) &= h_{\mathrm{D}}(\mathrm{d}_A(\emptyset, \{\langle 0, v\rangle\})) \\
&= h_{\mathrm{D}}(\{r \mid r <_{\mathbb{Q}} 0\}) \\
&= \{\{r \mid r <_{\mathbb{Q}} -\tfrac{1}{n+1}\} \mid n \in \mathbb{N} \setminus \{0\}\},
\end{aligned}
$$

by Proposition 2.8, for every $n \in \mathbb{N} \setminus \{0\}$, $h_{\mathrm{A}}(\{\langle 0, v\rangle\})$ is undefined at $\{r \mid r <_{\mathbb{Q}} -\tfrac{1}{n+1}\}$. Also, for every $n \in \mathbb{N}$,

$$
\begin{aligned}
h_{\mathrm{A}}(\{\langle 0, v\rangle\}) &= h_{\mathrm{A}}(\{\langle 0, v\rangle\} \sqcap_A \{\langle 0, v\rangle, \langle \tfrac{1}{n+1}, v\rangle\}) \\
&= h_{\mathrm{A}}(\{\langle 0, v\rangle\}) \sqcap_{\mathrm{S}[\langle T, \leq_T\rangle, V']} h_{\mathrm{A}}(\{\langle 0, v\rangle, \langle \tfrac{1}{n+1}, v\rangle\})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}_{\mathrm{S}[\langle T, \leq_T\rangle, V']}(h_{\mathrm{A}}(\{\langle 0, v\rangle\}), h_{\mathrm{A}}(\{\langle 0, v\rangle, \langle \tfrac{1}{n+1}, v\rangle\})) &= h_{\mathrm{D}}(\mathrm{d}_A(\emptyset, \{\langle 0, v\rangle, \langle \tfrac{1}{n+1}, v\rangle\})) \\
&= h_{\mathrm{D}}(\{r \mid r <_{\mathbb{Q}} \tfrac{1}{n+1}\}) \\
&= \{\{r \mid r <_{\mathbb{Q}} \tfrac{1}{n'+1}\} \mid n' \geq_{\mathbb{N}} n\},
\end{aligned}
$$

and thus, by Proposition 2.8, for any $n' <_{\mathbb{N}} n$, $h_{\mathrm{A}}(\{\langle 0, v\rangle\})$ is undefined at $\{r \mid r <_{\mathbb{Q}} \tfrac{1}{n'+1}\}$, and $h_{\mathrm{A}}(\{\langle 0, v\rangle\})$ is undefined at $\mathbb{Q}$. Thus, for every $n \in \mathbb{N}$, $h_{\mathrm{A}}(\{\langle 0, v\rangle\})$ is undefined at $\{r \mid r <_{\mathbb{Q}} \tfrac{1}{n+1}\}$, and $h_{\mathrm{A}}(\{\langle 0, v\rangle\})$ is undefined at $\mathbb{Q}$. Thus, $h_{\mathrm{A}}(\{\langle 0, v\rangle\}) = \emptyset$, and hence, for every $n \in \mathbb{N}$,

$$
\begin{aligned}
h_{\mathrm{A}}(\{\langle -\tfrac{1}{n+1}, v\rangle\}) \sqcap_{\mathrm{S}[\langle T, \leq_T\rangle, V']} h_{\mathrm{A}}(\{\langle 0, v\rangle\}) &= h_{\mathrm{A}}(\{\langle -\tfrac{1}{n+1}, v\rangle\}) \sqcap_{\mathrm{S}[\langle T, \leq_T\rangle, V']} \emptyset \\
&= \emptyset \\
&= h_{\mathrm{A}}(\{\langle 0, v\rangle\}),
\end{aligned}
$$

in contradiction to $\{h_{\mathrm{A}}, h_{\mathrm{D}}\}$ being an isomorphism between $\mathfrak{A}$ and $\mathfrak{S}[\langle T, \leq_T\rangle, V']$.

Therefore, for every totally ordered set $\langle T, \leq_T \rangle$ and every set V', there is no isomorphism between $\mathfrak{A}$ and an A-substructure of $\mathfrak{S}[\langle T, \leq_T\rangle, V']$.

The situation is illustrated in Figure 2. Figure 2(a) shows a Hasse diagram of the semilattice $\langle A, \sqcap_A \rangle$ of Example 4.9, where for every $n \in \mathbb{N}$, we let $s_{-(n+1)} = \{\langle -\tfrac{1}{n+1}, v\rangle\}$, we let $s_0 = \{\langle 0, v\rangle\}$, and for every $n \in \mathbb{N}$, we let $s_{n+1} = \{\langle 0, v\rangle, \langle \tfrac{1}{n+1}, v\rangle\}$. Figure 2(b) shows a Hasse diagram of the ordered set $\langle D, \supseteq_D \rangle$ of distances of Example 4.9, where the completely join-irreducible elements are shaded. These elements

---

[24] For every ordered set $\langle P, \leqslant \rangle$, and any $p \in P$, p is *completely join-irreducible* (also called *supercompact*) in $\langle P, \leqslant \rangle$ if and only if for every $C \subseteq P$, if C has a least upper bound $\bigvee C$ in $\langle P, \leqslant \rangle$, and $p \leqslant \bigvee C$, then there is $p' \in C$ such that $p \leqslant p'$.
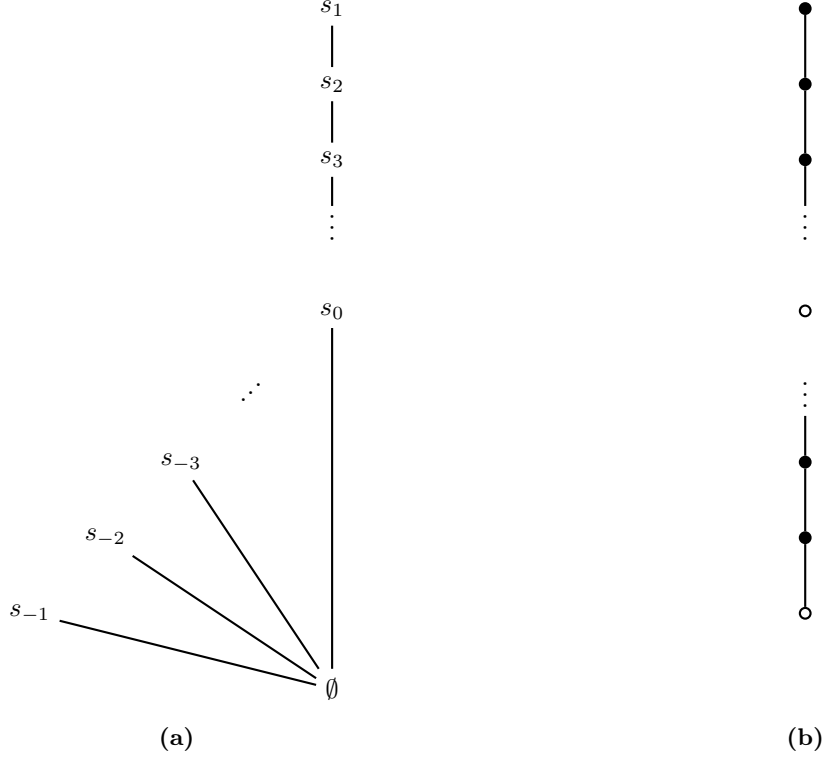
Figure 2. (a) A Hasse diagram of the semilattice $\langle A, \sqcap_A \rangle$ of Example 4.9, where for every $n \in \mathbb{N}$, we let $s_{-(n+1)} = \{\langle -\frac{1}{n+1}, v \rangle\}$, we let $s_0 = \{\langle 0, v \rangle\}$, and for every $n \in \mathbb{N}$, we let $s_{n+1} = \{\langle 0, v \rangle, \langle \frac{1}{n+1}, v \rangle\}$. (b) A Hasse diagram of the ordered set $\langle D, \supseteq_D \rangle$ of distances of Example 4.9, where the completely join-irreducible elements are shaded.

constitute the unique, up to order-isomorphism, candidate tag set for the representation of the generalized ultrametric semilattice $\mathfrak{A}$ of Example 4.9 as a structure in $\mathsf{S}_A(\mathcal{S}_{\mathrm{lin}})$. A signal corresponding to s_0 in such a structure could not be defined at any of the shaded elements in the lower half of the diagram of Figure 2(b), for there would not be enough shaded elements below that to accommodate all the distances between s_0 and the signals that are neither below nor above s_0 in Figure 2(a). And it could not be defined at any of the shaded elements in the upper half of the diagram of Figure 2(b), for there would not be enough shaded elements above that to accommodate all the signals that are above s_0 in the diagram of Figure 2(a).

Example 4.9 shows that the additional constraint that the ordered set of distances be order-isomorphic to the set of all lower sets of some totally ordered set ordered by inverse inclusion is, by itself, not sufficient for a generalized ultrametric semilattice to be representable as a structure in $\mathsf{S}_A(\mathcal{S}_{\mathrm{lin}})$. One might still be able to constrain generalized ultrametric semilattices with totally ordered distance sets further, and achieve a representation of that kind. But there are also good reasons not to go down that path. First, such a constraint would likely be unduly complicated and not first-order. And second, the related theory would be too specialized, inapplicable, for example, to structures of ordinary ultrametric spaces (e.g., see Example 3.10 and 3.11).

5 Axiomatization

We will now formalize the argument outlined in the beginning of Section 4. Throughout this section, we will assume some familiarity with first-order logic. Specifically, we will make reference to the standard

Table 1. Axioms of the theory of generalized ultrametric semilattices.

$\forall_A \alpha_1 \, \forall_A \alpha_2 \, \forall_A \alpha_3$	$(\alpha_1 \sqcap \alpha_2) \sqcap \alpha_3 = \alpha_1 \sqcap (\alpha_2 \sqcap \alpha_3)$	(S1)
$\forall_A \alpha_1 \, \forall_A \alpha_2$	$\alpha_1 \sqcap \alpha_2 = \alpha_2 \sqcap \alpha_1$	(S2)
$\forall_A \alpha$	$\alpha \sqcap \alpha = \alpha$	(S3)
$\forall_D \delta$	$\delta \leq \delta$	(P1)
$\forall_D \delta_1 \, \forall_D \delta_2 \, \forall_D \delta_3$	$(\delta_1 \leq \delta_2 \wedge \delta_2 \leq \delta_3 \rightarrow \delta_1 \leq \delta_3)$	(P2)
$\forall_D \delta_1 \, \forall_D \delta_2$	$(\delta_1 \leq \delta_2 \wedge \delta_2 \leq \delta_1 \rightarrow \delta_1 = \delta_2)$	(P3)
$\forall_D \delta$	$\mathbf{0} \leq \delta$	(P4)
$\forall_A \alpha$	$\mathbf{d}(\alpha, \alpha) = \mathbf{0}$	(G1)
$\forall_A \alpha_1 \, \forall_A \alpha_2$	$\mathbf{d}(\alpha_1, \alpha_2) = \mathbf{d}(\alpha_2, \alpha_1)$	(G2)
$\forall_A \alpha_1 \, \forall_A \alpha_2 \, \forall_A \alpha_3 \, \forall_D \delta$	$(\mathbf{d}(\alpha_1, \alpha_2) \leq \delta \wedge \mathbf{d}(\alpha_2, \alpha_3) \leq \delta \rightarrow \mathbf{d}(\alpha_1, \alpha_3) \leq \delta)$	(G3)
$\forall_A \alpha_1 \, \forall_A \alpha_2 \, \forall_A \alpha_3$	$(\mathbf{d}(\alpha_1, \alpha_2) \leq \mathbf{d}(\alpha_1, \alpha_3) \rightarrow (\alpha_1 \sqcap \alpha_3) \sqcap (\alpha_1 \sqcap \alpha_2) = \alpha_1 \sqcap \alpha_3)$	(A1)
$\forall_A \alpha_1 \, \forall_A \alpha_2 \, \forall_A \alpha_3$	$\mathbf{d}(\alpha_1 \sqcap \alpha_2, \alpha_1 \sqcap \alpha_3) \leq \mathbf{d}(\alpha_2, \alpha_3)$	(A2)

concepts of language, satisfaction, truth, model, deducibility, etc., for many-sorted first-order logic, which are, of course, straightforward generalizations of those for one-sorted first-order logic (e.g., see [9, chap. 4.3]). Our notation and terminology will, for the most part, be based on [9].

We write L for the first-order language of Σ.

For every class $\mathcal{C}$ of Σ-structures, we write Th$\mathcal{C}$ for the set of all sentences of L that are true in every member of $\mathcal{C}$.

We call Th$\mathsf{S}(\mathcal{S})$ the *theory of generalized ultrametric semilattices of signals.*

We call Th$\mathsf{S}(\mathcal{S}_{\mathrm{lin}})$ the *theory of generalized ultrametric semilattices of linear signals.*

Our main goal in this section is to axiomatize the theory of generalized ultrametric semilattices of linear signals.

We write GUS for the set of sentences of L listed in Table 1.

Sentences S1, S2, and S3 correspond to the standard axioms for semilattices, P1, P2, P3, and P4 to those for pointed ordered sets, and G1, G2, and G3 to those for generalized ultrametric spaces. Sentences A1 and A2 correspond to the two clauses of Proposition 2.9. Altogether, these sentences constitute a formal counterpart of Definition 3.7.

Proposition 5.1. $\mathfrak{A}$ *is a generalized ultrametric semilattice if and only if* $\mathfrak{A}$ *is a model of* GUS.

We write GUST for the set of all sentences of L that are deducible from GUS.

We call GUST the *theory of generalized ultrametric semilattices.*

The axioms of the theory of generalized ultrametric semilattices, namely the sentences in GUS, place no restriction on the distance set of a generalized ultrametric semilattice. We will need one more axiom to formalize the requirement that it be totally ordered.

We write T for the following sentence of L:

$$\forall_D \delta_1 \, \forall_D \delta_2 \, (\delta_1 \leq \delta_2 \vee \delta_2 \leq \delta_1).$$

By Proposition 3.9 and 5.1, GUS $\cup \{T\}$ is consistent. And clearly, S1, S2, S3, P1, P2, P3, P4, G1, G2, G3, and T constitute an independent set of sentences of L. It also follows from Example 2.10 (see also proof of

29

Figure 3. (a) A Hasse diagram of the semilattice $\langle |\mathfrak{A}|_A, \sqcap^{\mathfrak{A}} \rangle$ of Example 5.2. (b) A Hasse diagram of the pointed totally ordered set $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}} \rangle$ of Example 5.2.

[25, prop. 2.15.1]) that A2 is not deducible from all other sentences in $\mathrm{GUS} \cup \{T\}$. The following shows that A1 is not deducible from all other sentences in $\mathrm{GUS} \cup \{T\}$ either, establishing that $\mathrm{GUS} \cup \{T\}$ is actually independent:

Example 5.2. Let $\mathfrak{A}$ be a Σ-structure such that the following are true:

1. $\langle |\mathfrak{A}|_A, \sqcap^{\mathfrak{A}} \rangle$ is a semilattice defined by the Hasse diagram of Figure 3(a);

2. $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}} \rangle$ is a pointed totally ordered set defined by the Hasse diagram of Figure 3(b);

3. $\langle |\mathfrak{A}|_A, |\mathfrak{A}|_D, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}}, \mathbf{d}^{\mathfrak{A}} \rangle$ is a generalized ultrametric space such that the following are true:

   - $\mathbf{d}^{\mathfrak{A}}(a_0, a_1) = d_1$;
   - $\mathbf{d}^{\mathfrak{A}}(a_0, a_2) = d_2$;
   - $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) = d_2$.

It is easy to verify that every sentence in $\mathrm{GUS} \cup \{T\}$ except for A1 is true in $\mathfrak{A}$. However, A1 is not true in $\mathfrak{A}$; for

$$
\begin{aligned}
(a_1 \sqcap^{\mathfrak{A}} a_2) \sqcap^{\mathfrak{A}} (a_1 \sqcap^{\mathfrak{A}} a_0) &= a_0 \\
&\neq a_1 \\
&= a_1 \sqcap^{\mathfrak{A}} a_2,
\end{aligned}
$$

whereas

$$
\mathbf{d}^{\mathfrak{A}}(a_1, a_0) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_1, a_2).
$$

Note 5.1. The following Python script verifies that every sentence in $\mathrm{GUS} \cup \{T\}$ except for A1 is true in the structure $\mathfrak{A}$ of Example 5.2:

```
cardinality_of_A = 3

meet_operation_on_A = {(0, 0) : 0, (0, 1) : 0, (0, 2) : 0,
                       (1, 0) : 0, (1, 1) : 1, (1, 2) : 1,
                       (2, 0) : 0, (2, 1) : 1, (2, 2) : 2}

distance_function_on_A = {(0, 0) : 0, (0, 1) : 1, (0, 2) : 2,
                          (1, 0) : 1, (1, 1) : 0, (1, 2) : 2,
                          (2, 0) : 2, (2, 1) : 2, (2, 2) : 0}

def is_semilattice(meet):
    for a_1 in range(cardinality_of_A):
        for a_2 in range(cardinality_of_A):
            for a_3 in range(cardinality_of_A):
```

```python
                    if meet[(meet[(a_1, a_2)], a_3)] != meet[(a_1, meet[(a_2, a_3)])]:
                        print('The operation is not associative.')
                        return False
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                if meet[(a_1, a_2)] != meet[(a_2, a_1)]:
                    print('The operation is not commutative.')
                    return False
        for a in range(cardinality_of_A):
            if meet[(a, a)] != a:
                print('The operation is not idempotent.')
                return False
        return True

    def is_distance_function(distance):
        for a in range(cardinality_of_A):
            if distance[(a, a)] != 0:
                print('The function does not satisfy the identity of indiscernibles.')
                return False
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                if distance[(a_1, a_2)] != distance[(a_2, a_1)]:
                    print('The function does not satisfy symmetry.')
                    return False
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                for a_3 in range(cardinality_of_A):
                    if (distance[(a_1, a_2)] >
                            max(distance[(a_1, a_3)], distance[(a_3, a_2)])):
                        print('The function does not satisfy the generalized ' +
                            'ultrametric inequality.')
                        return False
        return True

    def models_GUS(meet, distance):
        if is_semilattice(meet) and is_distance_function(distance):
            print('The structure models GUS.')

    def models_A1(meet, distance):
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                for a_3 in range(cardinality_of_A):
                    if (distance[(a_1, a_2)] <= distance[(a_1, a_3)] and
                            meet[(meet[(a_1, a_3)], meet[(a_1, a_2)])] !=
                                meet[(a_1, a_3)]):
                        print('The structure does not model (A1): ' +
                            'a_1 = ' + str(a_1) +
                            ', a_2 = ' + str(a_2) +
                            ', a_3 = ' + str(a_3) +
                            '.')
                        return
        print('The structure models (A1).')

    def models_A2(meet, distance):
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                for a_3 in range(cardinality_of_A):
                    if (distance[(meet[(a_1, a_2)], meet[(a_1, a_3)])] >
                            distance[(a_2, a_3)]):
                        print('The structure does not model (A2).')
                        return
        print('The structure models (A2).')

    models_GUS(meet_operation_on_A, distance_function_on_A)
```

```
    models_A1(meet_operation_on_A, distance_function_on_A)

    models_A2(meet_operation_on_A, distance_function_on_A)
```

Running this script produces the following output:

```
The structure models GUS.
The structure does not model (A1): a_1 = 1, a_2 = 0, a_3 = 2.
The structure models (A2).
```

Notice that we have used the numerals 0, 1, and 2, with their standard ordering, to represent the totally ordered distance set of $\mathfrak{A}$.

We write $\text{GUST}_{\text{TODS}}$ for the set of all sentences of L that are deducible from $\text{GUS} \cup \{T\}$.

We call $\text{GUST}_{\text{TODS}}$ the *theory of generalized ultrametric semilattices with totally ordered distance sets*.

Theorem 5.3. $\text{Th}\,\mathsf{S}(\mathcal{S}_{\text{lin}}) = \text{GUST}_{\text{TODS}}$.

Proof. Assume a sentence σ of L.

Suppose that $\sigma \in \text{Th}\,\mathsf{S}(\mathcal{S}_{\text{lin}})$.

By Theorem 4.7, every model of $\text{GUST}_{\text{TODS}}$ is isomorphic to a structure in $\mathsf{S}(\mathcal{S}_{\text{lin}})$, and thus, since $\sigma \in \text{Th}\,\mathsf{S}(\mathcal{S}_{\text{lin}})$, a model of σ. Thus, by Gödel's completeness theorem, $\sigma \in \text{GUST}_{\text{TODS}}$.

Conversely, suppose that $\sigma \in \text{GUST}_{\text{TODS}}$.

By Proposition 3.9 and 3.8, every structure in $\mathsf{S}(\mathcal{S}_{\text{lin}})$ is a model of $\text{GUST}_{\text{TODS}}$, and thus, since $\sigma \in \text{GUST}_{\text{TODS}}$, a model of σ. Thus, by the soundness theorem for first-order logic, $\sigma \in \text{Th}\,\mathsf{S}(\mathcal{S}_{\text{lin}})$.

Thus, by generalization, $\text{Th}\,\mathsf{S}(\mathcal{S}_{\text{lin}}) = \text{GUST}_{\text{TODS}}$. $\qquad\square$

By Theorem 5.3, and the preceding discussion, $\text{GUS} \cup \{T\}$ is an independent axiomatization of the theory of generalized ultrametric semilattices of linear signals.

Now, we would like to address the issue raised at the end of Section 4 once more, this time from a logical point of view. In logical terms, the issue may be framed as whether it is the theory of structures in $\mathsf{S}_{\text{A}}(\mathcal{S}_{\text{lin}})$, rather than the theory of structures in $\mathsf{S}(\mathcal{S}_{\text{lin}})$, that one ought to be interested in. Clearly, $\mathsf{S}_{\text{A}}(\mathcal{S}_{\text{lin}})$ is a proper subclass of $\mathsf{S}(\mathcal{S}_{\text{lin}})$, and thus, it is natural to expect that there be sentences of L that are true in every structure in $\mathsf{S}_{\text{A}}(\mathcal{S}_{\text{lin}})$, but not in every structure in $\mathsf{S}(\mathcal{S}_{\text{lin}})$. Indeed, one such sentence is the sentence

$$\exists_{\text{D}}\, \delta_1 \ \forall_{\text{D}}\, \delta_2 \ \ \delta_2 \preceq \delta_1,$$

asserting the existence of a greatest element in the ordered set of distances, which would correspond to the empty lower set of the tag set in a structure in $\mathsf{S}_{\text{A}}(\mathcal{S}_{\text{lin}})$. But after some reflection, it appears that, at the end of the day, all such sentences express facts about the ordered set of distances alone, and not about the relationship between the meet operation and the generalized distance function in those structures. In an attempt to make this argument formal, we will now confine ourselves to a fragment of L that is free of variables, and hence, quantifiers, of sort D.

We write L_{A} for $\{\varphi \mid \varphi$ is a formula of L, and for every variable δ of sort D, δ does not occur in $\varphi\}$.

The following is straightforward:

Proposition 5.4. *If $\{h_{\text{A}}, h_{\text{D}}\}$ is an embedding of $\mathfrak{A}_1$ into $\mathfrak{A}_2$, and h_{A} is a one-to-one correspondence between $|\mathfrak{A}_1|_{\text{A}}$ and $|\mathfrak{A}_2|_{\text{A}}$, then for every formula φ of L_{A}, and every variable assignment $\{a_{\text{A}}, a_{\text{D}}\}$, $\mathfrak{A}_1$ satisfies φ with $\{a_{\text{A}}, a_{\text{D}}\}$ if and only if $\mathfrak{A}_2$ satisfies φ with $\{h_{\text{A}} \circ a_{\text{A}}, h_{\text{D}} \circ a_{\text{D}}\}$.*

Theorem 5.5. $\mathrm{Th}\,\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}}) \cap \mathrm{L}_{\mathrm{A}} = \mathrm{GUST}_{\mathrm{TODS}} \cap \mathrm{L}_{\mathrm{A}}$.

Proof. Assume a sentence σ of L_{A}.

Suppose that $\sigma \in \mathrm{Th}\,\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}})$.

Assume a model $\mathfrak{A}$ of $\mathrm{GUST}_{\mathrm{TODS}}$.

By Theorem 4.6 and Proposition 3.6, there is an A-substructure $\mathfrak{A}'$ of $\mathfrak{S}[\mathfrak{A}]$ such that $\{h_{\mathrm{A}}^{\mathfrak{A}}, h_{\mathrm{D}}^{\mathfrak{A}}\}$ is an embedding of $\mathfrak{A}$ into $\mathfrak{A}'$, and h_{A} is a one-to-one correspondence between $|\mathfrak{A}|_{\mathrm{A}}$ and $|\mathfrak{A}'|_{\mathrm{A}}$. Since $\mathfrak{A}' \in \mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}})$, $\mathfrak{A}'$ is a model of σ, and thus, by Proposition 5.4, $\mathfrak{A}$ is a model of σ .

Thus, by generalization and Gödel's completeness theorem, $\sigma \in \mathrm{GUST}_{\mathrm{TODS}}$.

Conversely, suppose that $\sigma \in \mathrm{GUST}_{\mathrm{TODS}}$.

By Proposition 3.9 and 3.8, every structure in $\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}})$ is a model of $\mathrm{GUST}_{\mathrm{TODS}}$, and thus, since $\sigma \in \mathrm{GUST}_{\mathrm{TODS}}$, a model of σ. Thus, by the soundness theorem for first-order logic, $\sigma \in \mathrm{Th}\,\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}})$.

Thus, by generalization, $\mathrm{Th}\,\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}}) \cap \mathrm{L}_{\mathrm{A}} = \mathrm{GUST}_{\mathrm{TODS}} \cap \mathrm{L}_{\mathrm{A}}$. $\qquad\square$

The following is immediate from Theorem 5.3 and 5.5:

Corollary 5.6. $\mathrm{Th}\,\mathsf{S}(\mathcal{S}_{\mathrm{lin}}) \cap \mathrm{L}_{\mathrm{A}} = \mathrm{Th}\,\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}}) \cap \mathrm{L}_{\mathrm{A}}$.

By Corollary 5.6, the relationship between the ordering among the signals and the distances between them in structures in $\mathsf{S}(\mathcal{S}_{\mathrm{lin}})$ is, as a whole, the same as that in structures in $\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}})$. The particular form that the ordered set of distances has in a structure in $\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}})$ is of no consequence to the first-order theory of that relationship in such structures. It is, for example, irrelevant whether or not the distance between any two signals is completely join-irreducible in the ordered set of distances, corresponding to a lower set of the tag set that has a greatest element, as opposed to one that does not. We believe that this, along with the argument outlined at the end of Section 4, justifies our interest in the theory of structures in $\mathsf{S}(\mathcal{S}_{\mathrm{lin}})$, rather than the theory of structures in $\mathsf{S}_{\mathrm{A}}(\mathcal{S}_{\mathrm{lin}})$.

Finally, it is natural to ask whether GUS axiomatizes the theory of generalized ultrametric semilattices of arbitrary signals, or in other words, whether $\mathrm{Th}\,\mathsf{S}(\mathcal{S}) = \mathrm{GUST}$. The answer is, unfortunately, no.

Example 5.7. Let σ be the following sentence of L:

$$\forall_{\mathrm{A}}\,\alpha_1 \; \forall_{\mathrm{A}}\,\alpha_2 \; \forall_{\mathrm{A}}\,\alpha_3 \; \forall_{\mathrm{A}}\,\alpha_4 \; (\mathbf{d}(\alpha_1, \alpha_2) \leq \mathbf{d}(\alpha_3, \alpha_4) \rightarrow (\alpha_3 \sqcap \alpha_4) \sqcap \alpha_1 = (\alpha_3 \sqcap \alpha_4) \sqcap \alpha_2).$$

It is easy to see that σ is true in every generalized ultrametric semilattice of signals; for every ordered set $\langle T, \leq_T \rangle$, every non-empty set V, and every $s_1, s_2, s_3, s_4 \in \mathrm{S}[\langle T, \leq_T \rangle, V]$, if

$$\mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_2) \supseteq \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_3, s_4),$$

then, by Proposition 2.8,

$$\mathsf{dom}(s_3 \sqcap_{\mathrm{S}[\langle T, \leq_T \rangle, V]} s_4) \subseteq \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_3, s_4)$$
$$\subseteq \mathrm{d}_{\mathrm{S}[\langle T, \leq_T \rangle, V]}(s_1, s_2),$$
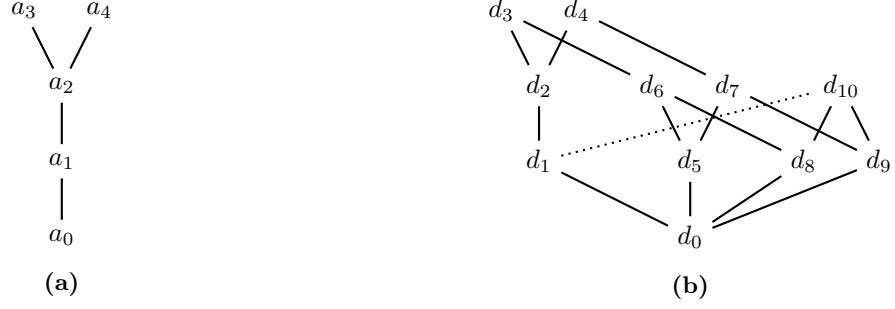
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(a)

(b)

Figure 4. (a) A Hasse diagram of the semilattice $\langle |\mathfrak{A}|_A, \sqcap^{\mathfrak{A}} \rangle$ of Example 5.7. (b) A Hasse diagram of the pointed ordered set $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}} \rangle$ of Example 5.7.

and thus, by Proposition 2.8,

$$
\begin{aligned}
(s_3 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_4) \sqcap_{S[\langle T, \leq_T \rangle, V]} s_1 &= ((s_3 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_4) \sqcap_{S[\langle T, \leq_T \rangle, V]} s_1) \upharpoonright d_{S[\langle T, \leq_T \rangle, V]}(s_1, s_2) \\
&= (s_3 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_4) \sqcap_{S[\langle T, \leq_T \rangle, V]} (s_1 \upharpoonright d_{S[\langle T, \leq_T \rangle, V]}(s_1, s_2)) \\
&= (s_3 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_4) \sqcap_{S[\langle T, \leq_T \rangle, V]} (s_1 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_2) \\
&= (s_3 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_4) \sqcap_{S[\langle T, \leq_T \rangle, V]} (s_2 \upharpoonright d_{S[\langle T, \leq_T \rangle, V]}(s_1, s_2)) \\
&= ((s_3 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_4) \sqcap_{S[\langle T, \leq_T \rangle, V]} s_2) \upharpoonright d_{S[\langle T, \leq_T \rangle, V]}(s_1, s_2) \\
&= (s_3 \sqcap_{S[\langle T, \leq_T \rangle, V]} s_4) \sqcap_{S[\langle T, \leq_T \rangle, V]} s_2.
\end{aligned}
$$

However, σ is not true in every generalized ultrametric semilattice.

Let $\mathfrak{A}$ be a Σ-structure such that the following are true:

1. $\langle |\mathfrak{A}|_A, \sqcap^{\mathfrak{A}} \rangle$ is a semilattice defined by the Hasse diagram of Figure 4(a);

2. $\langle |\mathfrak{A}|_D, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}} \rangle$ is a pointed ordered set defined by the Hasse diagram of Figure 4(b);

3. $\langle |\mathfrak{A}|_A, |\mathfrak{A}|_D, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}}, \mathbf{d}^{\mathfrak{A}} \rangle$ is a generalized ultrametric space such that the following are true:

    - $\mathbf{d}^{\mathfrak{A}}(a_0, a_1) = d_1$;
    - $\mathbf{d}^{\mathfrak{A}}(a_0, a_2) = d_2$;
    - $\mathbf{d}^{\mathfrak{A}}(a_0, a_3) = d_3$;
    - $\mathbf{d}^{\mathfrak{A}}(a_0, a_4) = d_4$;
    - $\mathbf{d}^{\mathfrak{A}}(a_1, a_2) = d_5$;
    - $\mathbf{d}^{\mathfrak{A}}(a_1, a_3) = d_6$;
    - $\mathbf{d}^{\mathfrak{A}}(a_1, a_4) = d_7$;
    - $\mathbf{d}^{\mathfrak{A}}(a_2, a_3) = d_8$;
    - $\mathbf{d}^{\mathfrak{A}}(a_2, a_4) = d_9$;
    - $\mathbf{d}^{\mathfrak{A}}(a_3, a_4) = d_{10}$.

It is straightforward to verify that every sentence in GUS is true in $\mathfrak{A}$. However, σ is not true in $\mathfrak{A}$; for

$$
\begin{aligned}
(a_3 \sqcap^{\mathfrak{A}} a_4) \sqcap^{\mathfrak{A}} a_0 &= a_0 \\
&\neq a_1 \\
&= (a_3 \sqcap^{\mathfrak{A}} a_4) \sqcap^{\mathfrak{A}} a_1,
\end{aligned}
$$

whereas

$$\mathbf{d}^{\mathfrak{A}}(a_0, a_1) \leq^{\mathfrak{A}} \mathbf{d}^{\mathfrak{A}}(a_3, a_4).$$

Note 5.2. The following Python script verifies that every sentence in GUS is true in the structure $\mathfrak{A}$ of Example 5.7, whereas the sentence σ of Example 5.7 is not true in $\mathfrak{A}$:

```
cardinality_of_A = 5

meet_operation_on_A = {(0, 0) : 0, (0, 1) : 0, (0, 2) : 0, (0, 3) : 0, (0, 4) : 0,
                       (1, 0) : 0, (1, 1) : 1, (1, 2) : 1, (1, 3) : 1, (1, 4) : 1,
                       (2, 0) : 0, (2, 1) : 1, (2, 2) : 2, (2, 3) : 2, (2, 4) : 2,
                       (3, 0) : 0, (3, 1) : 1, (3, 2) : 2, (3, 3) : 3, (3, 4) : 2,
                       (4, 0) : 0, (4, 1) : 1, (4, 2) : 2, (4, 3) : 2, (4, 4) : 4}

cardinality_of_D = 11

order_on_D = {( 0,  0) : 1, ( 0,  1) : 1, ( 0,  2) : 1, ( 0,  3) : 1, ( 0,  4) : 1,
                            ( 0,  5) : 1, ( 0,  6) : 1, ( 0,  7) : 1, ( 0,  8) : 1,
                            ( 0,  9) : 1, ( 0, 10) : 1,
              ( 1,  0) : 0, ( 1,  1) : 1, ( 1,  2) : 1, ( 1,  3) : 1, ( 1,  4) : 1,
                            ( 1,  5) : 0, ( 1,  6) : 0, ( 1,  7) : 0, ( 1,  8) : 0,
                            ( 1,  9) : 0, ( 1, 10) : 1,
              ( 2,  0) : 0, ( 2,  1) : 0, ( 2,  2) : 1, ( 2,  3) : 1, ( 2,  4) : 1,
                            ( 2,  5) : 0, ( 2,  6) : 0, ( 2,  7) : 0, ( 2,  8) : 0,
                            ( 2,  9) : 0, ( 2, 10) : 0,
              ( 3,  0) : 0, ( 3,  1) : 0, ( 3,  2) : 0, ( 3,  3) : 1, ( 3,  4) : 0,
                            ( 3,  5) : 0, ( 3,  6) : 0, ( 3,  7) : 0, ( 3,  8) : 0,
                            ( 3,  9) : 0, ( 3, 10) : 0,
              ( 4,  0) : 0, ( 4,  1) : 0, ( 4,  2) : 0, ( 4,  3) : 0, ( 4,  4) : 1,
                            ( 4,  5) : 0, ( 4,  6) : 0, ( 4,  7) : 0, ( 4,  8) : 0,
                            ( 4,  9) : 0, ( 4, 10) : 0,
              ( 5,  0) : 0, ( 5,  1) : 0, ( 5,  2) : 1, ( 5,  3) : 1, ( 5,  4) : 1,
                            ( 5,  5) : 1, ( 5,  6) : 1, ( 5,  7) : 1, ( 5,  8) : 0,
                            ( 5,  9) : 0, ( 5, 10) : 0,
              ( 6,  0) : 0, ( 6,  1) : 0, ( 6,  2) : 0, ( 6,  3) : 1, ( 6,  4) : 0,
                            ( 6,  5) : 0, ( 6,  6) : 1, ( 6,  7) : 0, ( 6,  8) : 0,
                            ( 6,  9) : 0, ( 6, 10) : 0,
              ( 7,  0) : 0, ( 7,  1) : 0, ( 7,  2) : 0, ( 7,  3) : 0, ( 7,  4) : 1,
                            ( 7,  5) : 0, ( 7,  6) : 0, ( 7,  7) : 1, ( 7,  8) : 0,
                            ( 7,  9) : 0, ( 7, 10) : 0,
              ( 8,  0) : 0, ( 8,  1) : 0, ( 8,  2) : 0, ( 8,  3) : 1, ( 8,  4) : 0,
                            ( 8,  5) : 0, ( 8,  6) : 1, ( 8,  7) : 0, ( 8,  8) : 1,
                            ( 8,  9) : 0, ( 8, 10) : 1,
              ( 9,  0) : 0, ( 9,  1) : 0, ( 9,  2) : 0, ( 9,  3) : 0, ( 9,  4) : 1,
                            ( 9,  5) : 0, ( 9,  6) : 0, ( 9,  7) : 1, ( 9,  8) : 0,
                            ( 9,  9) : 1, ( 9, 10) : 1,
              (10,  0) : 0, (10,  1) : 0, (10,  2) : 0, (10,  3) : 0, (10,  4) : 0,
                            (10,  5) : 0, (10,  6) : 0, (10,  7) : 0, (10,  8) : 0,
                            (10,  9) : 0, (10, 10) : 1}

distance_function_on_A = {(0, 0) :   0, (0, 1) :   1, (0, 2) :   2, (0, 3) :   3,
                                      (0, 4) :   4,
                          (1, 0) :   1, (1, 1) :   0, (1, 2) :   5, (1, 3) :   6,
                                      (1, 4) :   7,
                          (2, 0) :   2, (2, 1) :   5, (2, 2) :   0, (2, 3) :   8,
                                      (2, 4) :   9,
                          (3, 0) :   3, (3, 1) :   6, (3, 2) :   8, (3, 3) :   0,
                                      (3, 4) : 10,
                          (4, 0) :   4, (4, 1) :   7, (4, 2) :   9, (4, 3) : 10,
                                      (4, 4) :   0}

def is_semilattice(meet):
    for a_1 in range(cardinality_of_A):
```

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```python
            for a_2 in range(cardinality_of_A):
                for a_3 in range(cardinality_of_A):
                    if meet[(meet[(a_1, a_2)], a_3)] != meet[(a_1, meet[(a_2, a_3)])]:
                        print('The operation is not associative.')
                        return False
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                if meet[(a_1, a_2)] != meet[(a_2, a_1)]:
                    print('The operation is not commutative.')
                    return False
        for a in range(cardinality_of_A):
            if meet[(a, a)] != a:
                print('The operation is not idempotent.')
                return False
        return True

    def is_pointed_ordered_set(order):
        for d in range(cardinality_of_D):
            if order[(d, d)] != 1:
                print('The relation is not reflexive.')
                return False
        for d_1 in range(cardinality_of_D):
            for d_2 in range(cardinality_of_D):
                for d_3 in range(cardinality_of_D):
                    if (order[(d_1, d_2)] == 1 and order[(d_2, d_3)] == 1 and
                            order[(d_1, d_3)] != 1):
                        print('The relation is not transitive.')
                        return False
        for d_1 in range(cardinality_of_D):
            for d_2 in range(cardinality_of_D):
                if order[(d_1, d_2)] == 1 and order[(d_2, d_1)] == 1 and d_1 != d_2:
                    print('The relation is not antisymmetric.')
                    return False
        for d in range(cardinality_of_D):
            if order[(0, d)] != 1:
                print('The relation is not pointed.')
                return False
        return True

    def is_distance_function(order, distance):
        for a in range(cardinality_of_A):
            if distance[(a, a)] != 0:
                print('The function does not satisfy the identity of indiscernibles.')
                return False
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                if distance[(a_1, a_2)] == 0 and a_1 != a_2:
                    print('The function does not satisfy the identity of indiscernibles.')
                    return False
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                if distance[(a_1, a_2)] != distance[(a_2, a_1)]:
                    print('The function does not satisfy symmetry: ')
                    return False
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                for a_3 in range(cardinality_of_A):
                    for d in range(cardinality_of_D):
                        if (order[(distance[(a_1, a_2)], d)] == 1 and
                                order[(distance[(a_2, a_3)], d)] == 1 and
                                order[(distance[(a_1, a_3)], d)] != 1):
                            print('The function does not satisfy the generalized ' +
                                    'ultrametric inequality.')
                            return False
```

```python
        return True

    def models_GUS(meet, order, distance):
        if (is_semilattice(meet) and is_pointed_ordered_set(order) and
                is_distance_function(order, distance)):
            print('The_structure_models_GUS.')

    def models_A1(meet, order, distance):
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                for a_3 in range(cardinality_of_A):
                    if (order[(distance[(a_1, a_2)], distance[(a_1, a_3)])] == 1 and
                            meet[(meet[(a_1, a_3)], meet[(a_1, a_2)])] !=
                                meet[(a_1, a_3)]):
                        print('The_structure_does_not_model_(A1).')
                        return
        print('The_structure_models_(A1).')

    def models_A2(meet, order, distance):
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                for a_3 in range(cardinality_of_A):
                    if order[(distance[(meet[(a_1, a_2)], meet[(a_1, a_3)])]],
                            distance[(a_2, a_3)])] != 1:
                        print('The_structure_does_not_model_(A2).')
                        return
        print('The_structure_models_(A2).')

    def models_sigma(meet, order, distance):
        for a_1 in range(cardinality_of_A):
            for a_2 in range(cardinality_of_A):
                for a_3 in range(cardinality_of_A):
                    for a_4 in range(cardinality_of_A):
                        if (order[(distance[(a_1, a_2)], distance[(a_3, a_4)])] == 1 and
                                meet[(meet[(a_3, a_4)], a_1)] !=
                                    meet[(meet[(a_3, a_4)], a_2)]):
                            print('The_structure_does_not_model_sigma:_' +
                                    'a_1_=_' + str(a_1) +
                                    ',_a_2_=_' + str(a_2) +
                                    ',_a_3_=_' + str(a_3) +
                                    ',_a_4_=_' + str(a_4) +
                                    '.')
                            return
        print('The_structure_models_sigma.')

    models_GUS(meet_operation_on_A, order_on_D, distance_function_on_A)

    models_A1(meet_operation_on_A, order_on_D, distance_function_on_A)

    models_A2(meet_operation_on_A, order_on_D, distance_function_on_A)

    models_sigma(meet_operation_on_A, order_on_D, distance_function_on_A)
```

Running this script produces the following output:

```
The structure models GUS.
The structure models (A1).
The structure models (A2).
The structure does not model sigma: a_1 = 0, a_2 = 1, a_3 = 3, a_4 = 4.
```

The generalized ultrametric semilattice $\mathfrak{A}$ of Example 5.7 is, we believe, the smallest model of GUS in which the sentence σ of Example 5.7 is not true. The Hasse diagram consisting of just the solid edges of Figure 4(b) defines the least constrained distance set for which the semilattice defined by the Hasse

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diagram of Figure 4(a) yields a model of GUS, and its structure relative to that of the Hasse diagram of Figure 4(a) is interesting in its own right. The dotted edge is the extra constraint that causes σ to fail in $\mathfrak{A}$.

6 Related work

Ordered sets and metric spaces, especially ultrametric spaces, have been used extensively in the study of programming-language semantics, and there have been quite a few attempts at relating what are ostensibly two very different approaches to the mathematical modelling of computation. These attempts can be broadly categorized into two main lines of research.

One line of research has aimed for a common generalization of ordered sets and metric spaces, mostly in the form of some kind of generalized metric space (e.g., see [15], [35], [29], [34], [10], [3]). The main concern in that line of research has been the "reconciliation" between two very different kinds of topologies: the topology most naturally associated with an ordered set, namely the Scott topology, is not Hausdorff whereas the standard topology of a metric space is.

The other main line of research has been concerned with embedding metric spaces into suitably constructed ordered sets (e.g., see [36], [2], [14], [7], [11], [13]). The goal there has been to make use of the natural notion of approximation associated with ordered sets in providing simple computational models for classical mathematical spaces (e.g., see [6]).

Perhaps the work most closely related to our work here is that presented in [23] and [1]. The interest there is in semantic domains that are naturally endowed with both an order and a metric, but the main concern is once again topology, and no attempt is made to formalize the relationship between the two. We should also note that although some of the domains of the kind considered in [23] and [1] are, in fact, generalized ultrametric semilattices, this is not true in general.

Note 6.1. The following shows that a domain of the kind considered in [23] and [1] need not be a generalized ultrametric semilattice, even if that domain is a semilattice and an ultrametric space:

Example 6.1.1. Let $\mathfrak{A}$ be a Σ-structure such that the following are true:

1. $\langle |\mathfrak{A}|_{\mathrm{A}}, \sqcap^{\mathfrak{A}} \rangle$ is a semilattice defined by the following Hasse diagram:

$$
\begin{array}{c}
a_3 \\
| \\
a_2 \\
| \\
a_1 \\
| \\
a_0
\end{array}
$$

2. $\langle |\mathfrak{A}|_{\mathrm{D}}, \leq^{\mathfrak{A}}, \mathbf{0}^{\mathfrak{A}} \rangle = \langle \mathbb{R}, \leq_{\mathbb{R}}, 0 \rangle$;

3. $\mathbf{d}^{\mathfrak{A}}$ is the ultrametric $d[\rho]$ on $|\mathfrak{A}|_{\mathrm{A}}$ induced by a finite length ρ on $\langle |\mathfrak{A}|_{\mathrm{A}}, \sqsubseteq^{\mathfrak{A}} \rangle$, in the sense of [23] and [1], such that the following are true:

   - $\rho(a_0) = 0$;
   - $\rho(a_1) = 1$;
   - $\rho(a_2) = 1$;
   - $\rho(a_3) = 1$.

It is easy to verify that A2 is true in $\mathfrak{A}$. However, A1 is not true in $\mathfrak{A}$; for

$$\mathbf{d}^{\mathfrak{A}}(a_3, a_1) = 1$$
$$= \mathbf{d}^{\mathfrak{A}}(a_3, a_2),$$

whereas

$$(a_3 \sqcap^{\mathfrak{A}} a_2) \sqcap^{\mathfrak{A}} (a_3 \sqcap^{\mathfrak{A}} a_1) = a_1$$
$$\neq a_2$$
$$= a_3 \sqcap^{\mathfrak{A}} a_1.$$

Thus, $\mathfrak{A}$ is not an ultrametric semilattice.

The reason that the structure $\mathfrak{A}$ of Example 6.1.1 is not a generalized ultrametric semilattice is basically that the finite length ρ assigns the same natural number to distinct but comparable members of $|\mathfrak{A}|_A$. In fact, it is not hard to see that for every ordered set $\langle A, \leq_A \rangle$, and any finite length ρ on $\langle A, \leq_A \rangle$ such that for any $a_1, a_2 \in A$, if $a_1 <_A a_2$, then $\rho(a_1) <_A \rho(a_2)$, if $\langle A, \leq_A \rangle$ is a semilattice, then $\wedge_A$ and $d[\rho]$ structure A into an ultrametric semilattice, where $\wedge_A$ is a binary operator on A such that for any $a_1, a_2 \in A$, $a_1 \wedge_A a_2$ is the greatest lower bound of a_1 and a_2 in $\langle A, \leq_A \rangle$, and $d[\rho]$ the ultrametric on A induced by ρ. The motivation behind the more liberal definition of length in [23] and [1] is the desire to accommodate domains like event structures and trees as well.

7   Conclusion

The purpose of this work was twofold: (1) to axiomatize the relationship between the prefix relation and the generalized distance function in subsemilattices of linear signals, and (2) to lay the foundations of a more general theory of generalized ultrametric semilattices. That this was more than a mere theoretical exercise is attested by the recent success in proving, for the first time, a constructive fixed-point theorem for strictly causal functions on standard generalized ultrametric semilattices of signals (see [25], [27]), which, in retrospect, may be viewed as an application of the fixed-point theory of strictly contracting functions on generalized ultrametric semilattices (see [26]).

It is perhaps important to note that the use of generalized ultrametric spaces, rather than standard ultrametric spaces, is not the result of some misguided aspiration to generality. For example, real-time signals do not form an ultrametric semilattice under the standard prefix relation and the Cantor metric (e.g., see [17], [16]), which is really a consequence of the lack of an order-isomorphism between the ordered set $\langle \mathscr{L} \langle \mathbb{R}, \leq_{\mathbb{R}} \rangle, \supseteq_{\mathscr{L} \langle \mathbb{R}, \leq_{\mathbb{R}} \rangle} \rangle$ of generalized distances and that of real distances (see also [20, sec. 5.2]). It is only when we confine ourselves to discrete-event real-time signals that we may specialize our more general theory (see Example 3.11).

There are two interesting directions for future work. The first one is the axiomatization of the relationship between the prefix relation and the generalized distance function in subsemilattices of arbitrary signals. It is not at all obvious how to augment GUS to imply all formal properties of that relationship, and since there is no immediate practical use for those structures, the problem seems, at the present time, to be mainly of academic interest. The second and more pressing one is the continued development of the theory of generalized ultrametric semilattices with totally ordered distance sets. We believe that these are the right structures for a domain-theoretic treatment of timed computation. But for a complete treatment, we need to make sure that these structures remain closed under products and functions spaces of interest, and understand their topological properties.

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